II. *On an Extended Form of the Index Symbol in the Calculus of Operations.*

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Received November 24,—Read December 22, 1859.

In the following paper I propose to resume more in detail the subject of a communication made to the Cambridge and Dublin Mathematical Journal (vol. viii. p. 25). The present investigations are restricted to the case of two variables.

§ 1. *The Theory of* ∇ *and* ∇_1.

Let

$$x\frac{d}{dx}+y\frac{d}{dy}=\nabla,$$

$$y\frac{d}{dx}+x\frac{d}{dy}=\nabla_1;$$

and let

$$x\frac{d'}{dx}+y\frac{d'}{dy}=\Xi,$$

$$y\frac{d'}{dx}+x\frac{d'}{dy}=\Xi_1,$$

where the accent indicates that in the combinations of Ξ, Ξ_1 the differentiations are to affect only the subject of operation, and not x or y, so far as they appear explicitly in the values of Ξ, Ξ_1. So that Ξ^2, $\Xi\,\Xi_1$, Ξ_1^2 are to be understood as follows:—

$$\Xi^2=x^2\frac{d^2}{dx^2}+2xy\frac{d^2}{dx\,dy}+y^2\frac{d^2}{dy^2},$$

$$\Xi\Xi_1=xy\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)+(x^2+y^2)\frac{d^2}{dx\,dy},$$

$$\Xi_1^2=y^2\frac{d^2}{dx^2}+2yx\frac{d^2}{dx\,dy}+x^2\frac{d^2}{dy^2},$$

and so on for higher powers.

It will then be found that

$$\Xi^2=\nabla^2-\nabla \quad =(\nabla-1)\nabla,$$

$$\Xi\Xi_1=\nabla\nabla_1-\nabla_1=(\nabla-1)\nabla_1,$$

$$\Xi_1^2=\nabla_1^2-\nabla \quad =\nabla_1^2-\nabla.$$

Similarly,

$$\begin{aligned}\Xi^3&=\nabla\Xi^2-2\Xi^2 &&=(\nabla-2)\Xi^2,\\ \Xi^2\Xi_1&=\nabla\Xi\Xi_1-2\Xi\Xi_1 &&=(\nabla-2)\Xi\Xi_1,\\ \Xi\,\Xi_1^2&=\nabla\Xi_1^2-2\Xi_1^2 &&=(\nabla-2)\Xi_1^2,\\ \Xi_1^3&=\nabla_1\Xi_1^2-2\Xi\Xi_1 &&=\nabla\Xi_1^2-2\Xi\Xi_1.\end{aligned}$$

And, generally,

$$\begin{aligned}\Xi^n&=(\nabla-n+1)\Xi^{n-1},\\ \Xi^{n-1}\Xi_1&=(\nabla-n+1)\Xi^{n-2}\Xi_1,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \Xi\Xi_1^{n-1}&=(\nabla-n+1)\Xi_1^{n-1},\\ \Xi_1^n&=\nabla_1\Xi_1^{n-1}-(n-1)\Xi\Xi_1^{n-2}.\end{aligned}$$

Before developing these expressions, the following formulæ deserve notice:—

$$\begin{aligned}(\Xi\pm\Xi_1)^2&=\nabla^2\pm2\nabla\nabla_1+\nabla_1^2-2\nabla\mp2\nabla_1\\ &=(\nabla\pm\nabla_1-2)(\nabla\pm\nabla_1).\end{aligned}$$

Similarly, by arranging the terms of the third order thus,

$$\begin{aligned}\Xi^3&=\nabla\Xi^2 && &&-2\Xi^2,\\ 3\Xi^2\Xi_1&=2\nabla\Xi\Xi_1+ &&\nabla_1\Xi^2 &&-4\Xi\Xi_1-2\Xi\Xi_1,\\ 3\Xi\,\Xi_1^2&=\nabla\Xi_1^2 &&+2\nabla_1\Xi\Xi_1 &&-2\Xi_1^2\quad-2\Xi_1^2\quad-2\Xi^2,\\ \Xi_1^3&= && \nabla_1\Xi_1^2 &&-2\Xi\Xi_1,\end{aligned}$$

we have

$$(\Xi\pm\Xi_1)^3=(\nabla\pm\nabla_1-4)(\nabla\pm\nabla_1-2)(\nabla\pm\nabla_1).$$

And generally,

$$\begin{aligned}\Xi^n&=\nabla\Xi^{n-1}-(n-1)\Xi^{n-1},\\ \frac{n}{1}\Xi^{n-1}\Xi_1&=\frac{n-1}{1}\nabla\Xi^{n-2}\Xi_1+\nabla_1\Xi^{n-1}-(n-1)\frac{n-1}{1}\Xi^{n-2}\Xi_1-(n-1)\Xi^{n-2}\Xi_1,\\ \frac{n(n-1)}{1.2}\Xi^{n-2}\Xi_1^2&=\frac{(n-1)(n-2)}{1.2}\nabla\Xi^{n-3}\Xi_1+\frac{n-1}{1}\nabla_1\Xi^{n-2}\Xi_1-(n-1)\frac{(n-1)(n-2)}{1.2}\Xi^{n-3}\Xi_1^2-\frac{(n-1)(n-2)}{1.2}\Xi^{n-3}\Xi_1^2-\frac{n-1}{1}\Xi^{n-1},\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}$$

Taking the sum of these, the coefficient of $\Xi^{n-p-1}\Xi_1^p$ on the right-hand side will be

$$\begin{aligned}&\frac{(n-1)(n-2)..(n-p)}{1.2..p}(\nabla+\nabla_1)-(n-1)\frac{(n-1)(n-2)..(n-p)}{1.2..p}-(n-p)\frac{(n-1)(n-2)..(n-p+1)}{1.2..(p-1)}\\ &\qquad-(p+1)\frac{(n-1)(n-2)..(n-p-1)}{1.2..(p+1)}\\ &=\frac{(n-1)(n-2)..(n-p)}{1.2..p}\left\{\nabla+\nabla_1-(n-1)-p-(n-p-1)\right\}\\ &=\frac{(n-1)(n-2)..(n-p)}{1.2..p}\left\{\nabla+\nabla_1-2(n-1)\right\}.\end{aligned}$$

So that, finally,

$$(\Xi\pm\Xi_1)^n=\{\nabla\pm\nabla_1-2(n-1)\}(\Xi\pm\Xi_1)^{n-1}$$
$$=\{\nabla\pm\nabla_1-2(n-1)\}\{\nabla\pm\nabla_1-2(n-2)\}\ldots\{\nabla\pm\nabla_1-2\}\{\nabla\pm\nabla_1\},$$

the upper or the lower sign being taken throughout. And since

$$(\Xi\pm\Xi_1)^n=(x\pm y)^n\left(\frac{d}{dx}\pm\frac{d}{dy}\right)^n,$$

$$(x\pm y)^n\left(\frac{d}{dx}\pm\frac{d}{dy}\right)^n=\left\{(x\pm y)\left(\frac{d}{dx}\pm\frac{d}{dy}\right)-2(n-1)\right\}\left\{(x\pm y)\left(\frac{d}{dx}\pm\frac{d}{dy}\right)-2(n-2)\right\}\ldots\left\{(x\pm y)\left(\frac{d}{dx}\pm\frac{d}{dy}\right)\right\},$$

an equation which might be established by other methods.

Returning to the expressions for $\Xi^p\Xi_1^q$, we have

$$\Xi^n=(\nabla-n+1)\ldots(\nabla-2)(\nabla-1)\nabla,$$
$$\Xi^{n-1}\Xi_1=(\nabla-n+1)\ldots(\nabla-2)(\nabla-1)\nabla_1,$$
$$\Xi^{n-2}\Xi_1^2=(\nabla-n+1)\ldots(\nabla-2)(\nabla_1^2-\nabla),$$
$$\Xi^{n-3}\Xi_1^3=(\nabla-n+1)\ldots(\nabla_1^2-3\nabla+2)\nabla_1,$$
$$\Xi^{n-4}\Xi_1^4=(\nabla-n+1)\ldots(\nabla_1^4-6\nabla\nabla_1^2+3\nabla^2+8\nabla_1^2-6\nabla),$$
$$\Xi^{n-5}\Xi_1^5=(\nabla-n+1)\ldots(\nabla_1^5-10\nabla_1^3\nabla+12\nabla_1^3-8\nabla_1^2\nabla+12\nabla_1\nabla^2+3\nabla^3-44\nabla_1\nabla-6\nabla_1^2+24\nabla_1),$$

. . . .

These expressions are, however, determinants: thus,

$$\Xi^2=\begin{vmatrix}\nabla & 1\\ \nabla & \nabla\end{vmatrix}\qquad \Xi\Xi_1=\begin{vmatrix}\nabla & 1\\ \nabla_1 & \nabla_1\end{vmatrix}\qquad \Xi_1^2=\begin{vmatrix}\nabla_1 & 1\\ \nabla & \nabla_1\end{vmatrix}$$

$$\Xi^3=\begin{vmatrix}\nabla & 2 & \cdot\\ \nabla & \nabla & 1\\ \nabla & \nabla & \nabla\end{vmatrix}\qquad \Xi^2\Xi_1=\begin{vmatrix}\nabla & 2 & \cdot\\ \nabla & \nabla & 1\\ \nabla_1 & \nabla_1 & \nabla_1\end{vmatrix}\qquad \Xi\Xi_1^2=\begin{vmatrix}\nabla & 2 & \cdot\\ \nabla_1 & \nabla_1 & 1\\ \nabla & \nabla & \nabla_1\end{vmatrix}\qquad \Xi_1^3=\begin{vmatrix}\nabla_1 & 2 & \cdot\\ \nabla & \nabla_1 & 1\\ \nabla_1 & \nabla & \nabla_1\end{vmatrix}$$

The corresponding formulæ for $n=4$ are easily deduced from these, as follows:—For Ξ^4, $\Xi^3\Xi_1$, $\Xi^2\Xi_1^2$, $\Xi\Xi_1^3$, add to each determinant a row (∇, 3, 0, 0), and a column having ∇ for its upper constituent, and for the remainder a repetition of the first column of the determinant to which it is added. For Ξ_1^4 the additional row is (∇_1, 3, 0, 0), and the additional column ∇_1 with a repetition of the first column of $\Xi\Xi_1^2$. The formulæ then are

$$\Xi^4=\begin{vmatrix}\nabla & 3 & \cdot & \cdot\\ \nabla & \nabla & 2 & \cdot\\ \nabla & \nabla & \nabla & 1\\ \nabla & \nabla & \nabla & \nabla\end{vmatrix}\quad \Xi^3\Xi_1=\begin{vmatrix}\nabla & 3 & \cdot & \cdot\\ \nabla & \nabla & 2 & \cdot\\ \nabla & \nabla & \nabla & 1\\ \nabla_1 & \nabla_1 & \nabla_1 & \nabla_1\end{vmatrix}\quad \Xi^2\Xi_1^2=\begin{vmatrix}\nabla & 3 & \cdot & \cdot\\ \nabla & \nabla & 2 & \cdot\\ \nabla_1 & \nabla_1 & \nabla_1 & 1\\ \nabla & \nabla & \nabla & \nabla_1\end{vmatrix}\quad \Xi\Xi_1^3=\begin{vmatrix}\nabla & 3 & \cdot & \cdot\\ \nabla_1 & \nabla_1 & 2 & \cdot\\ \nabla & \nabla & \nabla_1 & 1\\ \nabla_1 & \nabla_1 & \nabla & \nabla_1\end{vmatrix}\quad \Xi_1^4=\begin{vmatrix}\nabla_1 & 3 & \cdot & \cdot\\ \nabla & \nabla_1 & 2 & \cdot\\ \nabla_1 & \nabla & \nabla_1 & 1\\ \nabla & \nabla_1 & \nabla & \nabla_1\end{vmatrix}$$

And so also generally,

$$\Xi_1^i = \begin{vmatrix} \nabla_1 & i-1 & 0 & .. & 0 & 0 \\ \nabla & \nabla_1 & i-2 & .. & 0 & 0 \\ \nabla_1 & \nabla & \nabla_1 & .. & 0 & 0 \\ . & . & . & .. & . & . \\ \nabla_1 & \nabla & \nabla_1 & .. & \nabla_1 & 1 \\ \nabla & \nabla_1 & \nabla & .. & \nabla & \nabla_1 \end{vmatrix}, \text{ or } = \begin{vmatrix} \nabla_1 & i-1 & 0 & .. & 0 & 0 \\ \nabla & \nabla_1 & i-2 & .. & 0 & 0 \\ \nabla_1 & \nabla & \nabla_1 & .. & 0 & 0 \\ . & . & . & .. & . & . \\ \nabla & \nabla_1 & \nabla & .. & \nabla_1 & 1 \\ \nabla_1 & \nabla & \nabla_1 & .. & \nabla & \nabla_1 \end{vmatrix}$$

according as i is even or odd. And the expression for $\Xi^j\Xi_1^i$ is to be formed from this by adding the following j rows (each consisting of $(i+j)$ places), viz.

$$\begin{matrix} \nabla & i+j-1 & 0 & .. & 0 & 0 \\ \nabla & \nabla & i+j-2 & .. & 0 & 0 \\ . & . & . & .. & . & . \\ \nabla & \nabla & \nabla & .. & 0 & 0 \end{matrix}$$

and repeating the first column of Ξ_1^i j times, so as to complete the square.

By means of these formulæ the expression

$$a\Xi^n + \frac{n}{1} b\Xi^{n-1}\Xi_1 + .. = (a, b, .. \between \Xi, \Xi_1)^n$$

may be exhibited as a function of ∇, ∇_1.

There is, however, another way in which this transformation may be effected. Let α, β ; α_1, β_1 ; .. be the roots of the equation

$$(a, b, .. \between \Xi, \Xi_1)^n = 0$$

solved with respect to $\Xi : \Xi_1$. Then

$$\begin{aligned} (a, b, c \between \Xi, \Xi_1)^2 &= (\alpha_1\Xi + \beta_1\Xi_1)(\alpha\Xi + \beta\Xi_1) \\ &= (\nabla\nabla_1 \between \alpha_1\beta_1)(\alpha\Xi + \beta\Xi_1) - \alpha_1(\alpha\Xi + \beta\Xi_1) - \beta_1(\alpha\Xi_1 + \beta\Xi) \\ &= (\nabla\nabla_1 \between \alpha_1\beta_1)(\nabla\nabla_1 \between \alpha\beta) - (\nabla\nabla_1\nabla \between \alpha_1\beta_1 \between \alpha\beta). \end{aligned}$$

Similarly,

$$\begin{aligned} &(a\, b\, c\, d \between \Xi\Xi_1)^3 \\ &= (\alpha_2\Xi + \beta_2\Xi_1)(\alpha_1\Xi + \beta_1\Xi_1)(\alpha\Xi + \beta\Xi) \\ &= (\alpha_2\nabla + \beta_2\nabla_1)(\alpha_1\Xi + \beta_1\Xi_1)(\alpha\Xi + \beta\Xi_1) \\ &\qquad -2\alpha_2\ (\alpha_1\Xi + \beta_1\Xi_1)(\alpha\,\Xi + \beta\Xi_1) \\ &\qquad -\ \beta_2\ (\alpha_1\Xi_1 + \beta_1\Xi\,)(\alpha\,\Xi + \beta\Xi_1) \\ &\qquad -\ \beta_2\ (\alpha_1\Xi + \beta_1\Xi_1)(\alpha\,\Xi_1 + \beta\Xi) \end{aligned}$$

$$
\begin{aligned}
&=(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\alpha\beta)-(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\nabla\between\alpha_1\beta_1\between\alpha\beta)\\
&\qquad -2\alpha_2(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\alpha\beta)+2\alpha_2(\nabla\nabla_1\nabla\between\alpha_1\beta_1\between\alpha\beta)\\
&\qquad -\ \beta_2(\nabla\nabla_1\between\beta_1\alpha_1)(\nabla\nabla_1\between\alpha\beta)+\ \beta_2(\nabla_1\nabla\nabla_1\between\alpha_1\beta_1\between\alpha\beta)\\
&\qquad -\ \beta_2(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\beta\alpha)+\ \beta_2(\nabla_1\nabla\nabla_1\between\alpha_1\beta_1\between\alpha\beta)\\
&=(\nabla\nabla\between\alpha_2\beta_2)(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\alpha\beta)\\
&\quad -(\nabla\nabla_1\between\alpha\ \beta)\ (\nabla\nabla_1\nabla\between\alpha_1\beta_1\between\alpha_2\beta_2)\\
&\quad -(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\nabla\between\alpha_2\beta_2\between\alpha\ \beta)\\
&\quad -(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\nabla\between\alpha\ \beta\ \between\alpha_1\beta_1)\\
&\quad +2(\nabla\nabla_1\nabla\nabla_1\between\alpha_2\beta_2\between\alpha_1\beta_1\between\alpha\beta).
\end{aligned}
$$

The expanded form of this is

$$
\begin{aligned}
&(\alpha_2\nabla+\beta_2\nabla_1)(\alpha_1\nabla+\beta_1\nabla_1)(\alpha\nabla+\beta\nabla_1)\\
&-(\alpha\ \nabla+\beta\ \nabla_1)\{(\alpha_1\alpha_2+\beta_1\beta_2)\nabla+(\alpha_1\beta_2+\alpha_2\beta_1)\nabla_1\}\\
&-(\alpha_1\nabla+\beta_1\nabla_1)\{(\alpha_2\alpha\ +\beta_2\beta\ )\nabla+(\alpha_2\beta\ +\alpha\ \beta_2)\nabla_1\}\\
&-(\alpha_2\nabla+\beta_2\nabla_1)\{(\alpha\ \alpha_1+\beta\ \beta_1)\nabla+(\alpha\ \beta_1+\alpha_1\beta)\ \nabla_1\}\\
&\quad +2\{(\alpha\alpha_1\alpha_2+\alpha\beta_1\beta_2+\alpha_1\beta_2\beta+\alpha_2\beta\beta_1)\nabla\\
&\qquad +(\beta\beta_1\beta_2+\beta\alpha_1\alpha_2+\beta_1\alpha_2\alpha+\beta_2\alpha\alpha_1)\nabla_1\}\\
&=a\nabla^3+3b\nabla^2\nabla_1+3c\nabla\nabla_1^2+d\nabla_1^3\\
&\quad -3(a+c)\nabla^2-3(3b+d)\nabla\nabla_1-bc\nabla_1^2\\
&\qquad +2(a+3c)\nabla+2(3b+d)\nabla_1.
\end{aligned}
$$

To these may be added,

$$
\begin{aligned}
&(a\ b\ c\ d\ e\between\Xi\ \Xi_1)^4\\
&=(\nabla\nabla_1\between\alpha_3\beta_3)(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\alpha\beta)\\
&\quad -(\nabla\nabla_1\between\alpha_3\beta_3)(\nabla\nabla_1\between\alpha\ \beta\ )(\nabla\nabla_1\nabla\between\alpha_2\beta_2\between\alpha_1\beta_1)\\
&\quad -(\nabla\nabla_1\between\alpha_3\beta_3)(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\nabla\between\alpha\ \beta\ \between\alpha_2\beta_2)\\
&\quad -(\nabla\nabla_1\between\alpha_3\beta_3)(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\nabla\between\alpha_1\beta_1\between\alpha\ \beta)\\
&\quad -(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\nabla\between\alpha_3\beta_3\between\alpha\ \beta)\\
&\quad -(\nabla\nabla_1\between\alpha\ \beta\ )(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\nabla\between\alpha_3\beta_3\between\alpha_1\beta_1)\\
&\quad -(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\between\alpha\ \beta\ )(\nabla\nabla_1\nabla\between\alpha_3\beta_3\between\alpha_2\beta_2)\\
&\qquad +2(\nabla\nabla_1\between\alpha\ \beta\ )(\nabla\nabla_1\nabla\nabla_1\between\alpha_3\beta_3\between\alpha_2\beta_2\between\alpha_1\beta_1)\\
&\qquad +2(\nabla\nabla_1\between\alpha_1\beta_1)(\nabla\nabla_1\nabla\nabla_1\between\alpha\ \beta\ \between\alpha_3\beta_3\between\alpha_2\beta_2)\\
&\qquad +2(\nabla\nabla_1\between\alpha_2\beta_2)(\nabla\nabla_1\nabla\nabla_1\between\alpha_1\beta_1\between\alpha\ \beta\ \between\alpha_3\beta_3)\\
&\qquad +2(\nabla\nabla_1\between\alpha_3\beta_3)(\nabla\nabla_1\nabla\nabla_1\between\alpha_2\beta_2\between\alpha_1\beta_1\between\alpha\ \beta)\\
&\qquad +(\nabla\nabla_1\nabla\between\alpha_3\beta_3\between\alpha\ \beta\ )(\nabla\nabla_1\nabla\between\alpha_2\beta_2\between\alpha_1\beta_1)
\end{aligned}
$$

$$
\begin{aligned}
&+(\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_1\beta_1)(\nabla\nabla_1\nabla)(\alpha\ \beta\ )(\alpha_2\beta_2)\\
&+(\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_2\beta_2)(\nabla\nabla_1\nabla)(\alpha_1\beta_1)(\alpha\ \beta)\\
&-6(\nabla\nabla_1\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_2\beta_2)(\alpha_1\beta_1)(\alpha\beta)\\
&=a\nabla^4+4b\nabla^3\nabla_1+6c\nabla^2\nabla_1^2+4d\nabla\nabla_1^3+e\nabla_1^4\\
&-6\{(a+c)\nabla^3+2(2b+d)\nabla^2\nabla_1+(5c+e)\nabla\nabla_1^2+2d\nabla_1^3\}\\
&+(11a+30c+3d)\nabla^2+44(b+d)\nabla\nabla_1+4(6c+2d+3e)\nabla_1^2\\
&\qquad-6\{(a+6c+e)\nabla+4(b+d)\nabla_1\}.
\end{aligned}
$$

There is considerable resemblance between these functions and determinants; and although the factors of which the various terms are composed do not admit of being arranged as constituents of the determinants which they resemble, yet a symbolical notation will render such an arrangement possible. For this purpose, let $(\nabla\nabla_1)(\alpha_i\beta_i)($ and $)(\alpha_j\beta_j)$ be the symbolical factors of $(\nabla\nabla_1\nabla)(\alpha_i\beta_i)(\alpha_j\beta_j)$; so that

$$(\nabla\nabla_1)(\alpha_2\beta_2)(\times)(\alpha_1\beta_1)=(\nabla\nabla_1\nabla)(\alpha_2\beta_2)(\alpha_1\beta_1)$$

$$(\nabla\nabla_1)(\alpha_2\beta_2)(\times)(\alpha_1\beta_1)(\times)(\alpha\beta)=(\nabla\nabla_1\nabla\nabla_1)(\alpha_2\beta_2)(\alpha_1\beta_1)(\alpha\beta),$$

and so on. Then

$$
\begin{aligned}
&(a\ b\ c)(\Xi\Xi_1)^2\\
&=\begin{vmatrix}(\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha\beta)\\ (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta)\end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&(a\ b\ c)(\Xi\Xi_1)^3\\
&=\begin{vmatrix}(\nabla\nabla_1)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha\beta)\\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha\beta)\\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta)\end{vmatrix}
\end{aligned}
$$

By means of these formulæ, the corresponding expressions for higher degrees may be established. Thus, for the fourth degree,

$$
\begin{aligned}
&(a\ b\ c\ d\ e)(\Xi\ \Xi_1)^4\\
&=(\alpha_3\Xi+\beta_3\Xi_1)(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi)\\
&=(\alpha_3\nabla+\beta_3\nabla_1)(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\qquad-3\alpha_3(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\qquad-\ \beta_3(\beta_2\Xi+\alpha_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\qquad-\ \beta_3(\beta_2\Xi+\beta_2\Xi_1)(\beta_1\Xi+\alpha_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\qquad-\ \beta_3(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\beta\Xi+\alpha\Xi_1)\\
&=(\alpha_3\nabla+\beta_3\nabla_1)(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\quad-(\Xi\Xi_1\Xi)(\alpha_3\beta_3)(\alpha_2\beta_2)(\alpha_1\Xi+\beta_1\Xi_1)(\alpha\Xi+\beta\Xi_1)\\
&\quad-(\alpha_2\Xi+\beta_2\Xi_1)(\Xi\Xi_1\Xi)(\alpha_3\beta_3)(\alpha_1\beta_1)(\alpha\Xi+\beta\Xi_1)\\
&\quad-(\alpha_2\Xi+\beta_2\Xi_1)(\alpha_1\Xi+\beta_1\Xi_1)(\Xi\Xi\ \Xi)(\alpha_3\beta_3)(\alpha\beta)
\end{aligned}
$$

$$=(\nabla\nabla_1)(\alpha_3\beta_3)\begin{vmatrix} (\nabla\nabla_1)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta) \end{vmatrix}$$

$$-\begin{vmatrix} (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta) \end{vmatrix}$$

$$-\begin{vmatrix} (\nabla\nabla_1)(\alpha_2\beta_2) &)(\alpha_3\beta_3)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta) \end{vmatrix}$$

$$-\begin{vmatrix} (\nabla\nabla_1)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha_3\beta_3)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha_3\beta_3)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1\nabla)(\alpha_3\beta_3)(\alpha\beta) \end{vmatrix}$$

Transposing the columns of the third of these determinants, the sign becomes changed, and the sum

$$=\begin{vmatrix} (\nabla\nabla_1)(\alpha_3\beta_3) &)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_3\beta_3)(& (\nabla\nabla_1)(\alpha_2\beta_2) &)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_3\beta_3)(& (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1) &)(\alpha\beta) \\ (\nabla\nabla_1)(\alpha_3\beta_3)(& (\nabla\nabla_1)(\alpha_2\beta_2)(& (\nabla\nabla_1)(\alpha_1\beta_1)(& (\nabla\nabla_1)(\alpha\beta) \end{vmatrix}$$

And as the above process is perfectly general in principle, we may conclude the general expression

$$(a\,b\,..\,)(\Xi\Xi_1)^n$$

$$=\begin{vmatrix} (\nabla\nabla_1)(\alpha_n\beta_n) &)(\alpha_{n-1}\beta_{n-1}) & .. &)(\alpha_1\beta_1) \\ (\nabla\nabla_1)(\alpha_n\beta_n)(& (\nabla\nabla_1)(\alpha_{n-1}\beta_{n-1}) & .. &)(\alpha_1\beta_1) \\ . & . & .. & . \\ (\nabla\nabla_1)(\alpha_n\beta_n)(& (\nabla\nabla_1)(\alpha_{n-1}\beta_{n-1})(& .. & (\nabla\nabla_1)(\alpha_1\beta_1) \end{vmatrix}$$

In order to alculate the effect of the operation ∇_1, and so that of the operation $(a\,b\,..\,)(\Xi\,\Xi_1)^n$ upon a given function, let

$$u=a_0x^n+a_1x^{n-1}y+\,..$$
$$=\Sigma a_ix^{n-i}y^i.$$

Then

$$\nabla_1u=\Sigma a_i\{(n-i)x^{n-i-1}y^{i+1}+ix^{n-i+1}y^{i-1}\}$$
$$=\Sigma\{(n-i+1)a_{i-1}+(i+1)a_{i+1}\}x^{n-i}y^i.$$

Now

$$\varepsilon^{k\frac{d}{d\theta}}f(\theta)=f(\theta+k).$$

Hence

$$\nabla_1 n=\Sigma\left\{(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right\}a_i x^{n-i}y^i,$$

$$\nabla_1^2 u=\Sigma\left\{(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right\}^2 a_i x^{n-i}y^i,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\nabla_1^m u=\Sigma\left\{(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right\}^m a_i x^{n-i}y^i,$$

where the operative factor must be understood to affect the suffixes of the as and the is in the coefficients, and not the indices of the variables. In the use of these formulæ, however, it is to be observed that the order of the multiplications must be retained. Thus:

$$\begin{aligned}
&\left\{(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right\}^2 a_i\\
&=\Big\{(n-i+1)\varepsilon^{-\frac{d}{di}}(n-i+1)\varepsilon^{-\frac{d}{di}}\\
&\quad+(n-i+1)\varepsilon^{-\frac{d}{di}}(i+1)\varepsilon^{\frac{d}{di}}\\
&\quad+\quad(i+1)\varepsilon^{\frac{d}{di}}(n-i+1)\varepsilon^{-\frac{d}{di}}\\
&\quad+\quad(i+1)\varepsilon^{\frac{d}{di}}(i+1)\varepsilon^{\frac{d}{di}}\Big\}a_i
\end{aligned}$$

$$=\left\{(n-i+1)(n-i+2)\varepsilon^{-2\frac{d}{di}}+(n-i+1)i+(n-i)(i+1)+(i+1)(i+2)\varepsilon^{2\frac{d}{di}}\right\}a_i$$

$$=\ (n-i+1)(n-i+2)a_{i-2}+\left\{(n-i+1)i+(n-i)(i+1)\right\}a_i+(i+1)(i+2)a_{i-2}.$$

Similarly, since

$$\begin{aligned}
(a+b)^3 = {} & a\ a\ a\\
&+b\ a\ a\\
&+a\ b\ a\\
&+a\ a\ b\\
&+a\ b\ b\\
&+b\ a\ b\\
&+b\ b\ a\\
&+b\ b\ b.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\{(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right\}^3 a_i\\
&=\Big[(n-i+1)(n-i+2)(n-i+3)\varepsilon^{-3\frac{d}{di}}\\
&\quad+\left\{(i+1)(n-i)(n-i+1)+(n-i+1)i(n-i+1)+(n-i+1)(n-i+2)(i-1)\right\}\varepsilon^{-\frac{d}{di}}\\
&\quad+\left\{(n-i+1)i(i+1)+(i+1)(n-i)(i+1)+(i+1)(i+2)(n-i-1)\right\}\varepsilon^{\frac{d}{di}}\\
&\quad+\ (i+1)(i+2)(i+3)\varepsilon^{3\frac{d}{di}}\Big]a_i
\end{aligned}$$

$$= (n-i+1)(n-i+2)(n-i+3)a_{i-3}$$
$$+\{(n-i)(n-i+1)(i+1)+(n-i+1)^2 i+(n-i+1)(n-i+2)(i-1)\}a_{i-1}$$
$$+\{(n-i+1)i(i+1)+(n-i)(i+1)^2+(n-i-1)(i+1)(i+2)\}a_{i+1}$$
$$+ (i+1)(i+2)(i+3)a_{i+3},$$

and so on for higher degrees.

But it is also required to determine the effect of negative powers of ∇_1 on a given function. For this purpose let

$$U=\Sigma A_i x^{n-i} y^i$$

and

$$\nabla_1^{-1}U=u, \text{ or } U=\nabla_1 u.$$

Then equating coefficients,

$$\begin{array}{rcl} & 1a_1 & =A_0, \\ na_0 + & 2a_2 & =A , \\ (n-1)a_0 + & 3a_3 & =A_2, \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\ 3a_{n-3}+ & (n-1)a_{n-1} & =A_{n-2}, \\ 2a_{n-2}+ & na_n & =A_{n-1}, \\ a_{n-1} & & =A_n; \end{array}$$

whence, by actual elimination,

$$a_1=\frac{1}{1}A_0,$$

$$a_3=\frac{1}{3}A_2-\frac{n-1}{3}a_1$$

$$=\frac{1}{3}\left(A_2-\frac{n-1}{1}A_0\right),$$

$$a_5=\frac{1}{5}\left(A_4-(n-3)a_3\right)$$

$$=\frac{1}{5}\left(A_4-\frac{n-3}{3}A_2+\frac{(n-3)(n-1)}{3.1}A_0\right),$$

$$a_7=\frac{1}{7}\left(A_6-(n-5)a_5\right)$$

$$=\frac{1}{7}\left(A_6-\frac{n-5}{5}A_4+\frac{(n-5)(n-3)}{5.3}A_2-\frac{(n-5)(n-3)(n-1)}{5.3.1}A_0\right),$$

. . . .

$$a_{2m+1}=\frac{1}{2m+1}\left(A_{2m}-\frac{n-2m+1}{2m-1}A_{2m-2}+\frac{(n-2m+1)(n-2m+3)}{(2m-1)(2m-3)}A_{2m-4}-\ldots\right.$$
$$\left.(-)^m\frac{(n-2m+1)(n-2m+3)\ldots(n-1)}{(2m-1)(2m-3)\ldots 1}A_0\right);$$

and for the even indices,

$$a_2=\frac{1}{2}\Big(A_1-na_0\Big),$$

$$a_4=\frac{1}{4}\Big(A_3-(n-2)a_2\Big)$$

$$=\frac{1}{4}\Big(A_3-\frac{n-2}{2}A_1+\frac{(n-2)n}{2.1}a_0\Big),$$

$$a_6=\frac{1}{6}\Big(A_5-(n-4)a_4\Big)$$

$$=\frac{1}{6}\Big(A_5-\frac{n-4}{4}A_3+\frac{(n-4)(n-2)}{4.2}A_1-\frac{(n-4)(n-2)n}{4.2.1}a_0\Big),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

But it is better to make use of the operative symbol for the solution of the equations; thus:

$$(n-i+2)a_{i-2}+ia_i=A_{i-1}$$

$$\Big\{(n-i+2)\varepsilon^{-2\frac{d}{di}}+i\Big\}a_i=A_{i-1}, \text{ or } \Big\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}+1\Big\}ia_i=A_{i-1};$$

whence

$$a_i=\Big\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}+\Big\}^{-1}A_{i-1},$$

$$a_i=\frac{1}{i}\Big\{1+(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\Big\}^{-1}A_{i-1},$$

$$a_i=\frac{1}{i}\Big\{A_{i-1}-\frac{n-i+2}{i-2}A_{i-3}+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}-\ldots\Big\};$$

the last term being

$$(-)^{\frac{i-1}{2}}\frac{(n-i+2)(n-i+4)\ldots(n-1)}{(i-2)(i-4)\ldots1}A_0 \text{ when } i \text{ is odd,}$$

and

$$(-)^{\frac{i}{2}}\frac{(n-i+2)(n-i+4)\ldots n}{(i-2)(i-4)\ldots2}a_0 \text{ when } i \text{ is even.}$$

The two cases, however, of n even and n odd require special notice. If n be odd, the last equation of the series for determining the as may be thus written:

$$na_{n+1}=0=A_n-\frac{1}{n-1}A_{n-2}+\frac{1.3}{(n-1)(n-3)}A_{n-4}-\ldots(-)^{\frac{n+1}{2}}\frac{1.3\ldots n}{(n-1)(n-3)\ldots2}a_0,$$

which determines a_0.

But in the case of n being even,

$$na_{n+1}=0=A_n-\frac{1}{n-1}A_{n-2}+\frac{1.3}{(n-1)(n-3)}A_{n-4}-\ldots(-)^{\frac{n}{2}}A_0,$$

a relation between the As alone. The explanation of this apparent anomaly is to be sought in the complementary arbitrary function arising out of the operation $\nabla_1^{-1}0$.

In the case of $\nabla_1^2 u$,

$$a_i=\left\{(n-i+2)\varepsilon^{-2\frac{d}{di}}+i\right\}^2 A_{i-1}$$

$$=\frac{1}{i}\left\{1+(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^{-1}\frac{1}{i}\left\{1+(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^{-1}A_{i-1}$$

$$=\frac{1}{i}\left\{1+(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^{-1}\frac{1}{i}\left\{A_{i-1}-\frac{n-i+2}{i-2}A_{i-3}+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}-..\right\}$$

$$=\frac{1}{i}\left\{1-\left\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}+\left\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^2-..\right\}\frac{1}{i}\left\{A_{i-1}-\frac{n-i+2}{i-2}A_{i-3}+..\right\}.$$

But

$$\left\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^2=(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\frac{n-i+2}{i-2}\varepsilon^{-2\frac{d}{di}}$$

$$=\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\varepsilon^{-4\frac{d}{di}},$$

$$\left\{(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\right\}^3=(n-i+2)\varepsilon^{-2\frac{d}{di}}\frac{1}{i}\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\varepsilon^{-4\frac{d}{di}}$$

$$=\frac{(n-i+2)(n-i+4)(n-i+6)}{(i-2)(i-4)(i-6)}\varepsilon^{-6\frac{d}{di}}.$$

Therefore

$$\begin{aligned} ia_i=\frac{1}{i}\Big\{&A_{i-1}-\frac{n-i+2}{i-2}A_{i-3}+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}-..\Big\}\\ &-\frac{1}{i-2}\left\{\frac{n-i+2}{i-2}A_{i-3}-\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}+..\right\}\\ &\qquad+\frac{1}{i-4}\left\{\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}-..\right\}\\ &\qquad\qquad-\qquad\qquad ..\Big\} \end{aligned}$$

$$\begin{aligned} =\frac{1}{i}&A_{i-1}\\ &-\left(\frac{1}{i}+\frac{1}{i-2}\right)\frac{n-i+2}{i-2}A_{i-3}\\ &+\left(\frac{1}{i}+\frac{1}{i-2}+\frac{1}{i-4}\right)\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}A_{i-5}\\ &-\ldots, \end{aligned}$$

the last term being

$$(-)^{\frac{i-1}{2}}\left(\frac{1}{i}+\frac{1}{i-2}+..+\frac{1}{3}+1\right)\frac{(n-i+2)(n-i+4)..(n-1)}{(i-2)(i-4)..1}A_0 \text{ when } i \text{ is odd,}$$

and

$$(-)^{\frac{i}{2}}\left(\frac{1}{i}+\frac{1}{i-2}+..+\frac{1}{4}+\frac{1}{2}\right)\frac{(n-i+2)(n-i+4)..n}{(i-2)(i-4)..2}a_0 \text{ when } i \text{ is even.}$$

For $\nabla_1^3 u$,

$$ia_i=\left\{1-\frac{n-i+2}{i-2}\varepsilon^{-2\frac{d}{di}}+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\varepsilon^{-4\frac{d}{di}}-..\right\}\frac{1}{i}\left\{\frac{1}{i}A_{i-1}-\left(\frac{1}{i}+\frac{1}{i-2}\right)\frac{n-i+2}{i-2}A_{i-3}+..\right\}$$

$$=\frac{1}{i}\left\{\frac{1}{i}A_{i-1}-\left(\frac{1}{i}+\frac{1}{i-2}\right)\frac{n-i+2}{i-2}A_{i-3}+..\right\}$$

$$-\frac{n-i+2}{i-2}\cdot\frac{1}{(i-2)}\left\{\frac{1}{i-2}A_{i-3}-\left(\frac{1}{i-2}+\frac{1}{i-4}\right)\frac{n-i+4}{i-4}A_{i-5}+..\right\}$$

$$+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\frac{1}{i-4}\left\{\frac{1}{i-4}A_{i-5}-\left(\frac{1}{i-4}+\frac{1}{i-6}\right)\frac{n-i+6}{i-6}A_{i-7}+..\right\}$$

$$-\ldots$$

and writing for convenience $\left(\frac{1}{i},\frac{1}{i-2}..\right)^m=\left(\frac{1}{i}\right)^m+\left(\frac{1}{i}\right)^{m-1}\frac{1}{i-2}+..$, *i. e.* the multinomial expansion with the numerical coefficients suppressed,

$$ia_i=\left(\frac{1}{i}\right)^2A_{i-1}$$

$$-\frac{n-i+2}{i-2}\left(\frac{1}{i},\frac{1}{i-2}\right)^2A_{i-3}$$

$$+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\left(\frac{1}{i},\frac{1}{i-2},\frac{1}{i-4}\right)^2A_{i-5}$$

$$-\ldots$$

the last term being

$$(-)^{\frac{i-1}{2}}\frac{(n-i+2)(n-i+4)..(n-1)}{(i-2)(i-4)..1}\left(\frac{1}{i},\frac{1}{i-2},..\frac{1}{3},1\right)^2A_0 \text{ when } i \text{ is odd,}$$

and

$$(-)^{\frac{i}{2}}\frac{(n-i+2)(n-i+4)..n}{(i-2)(i-4)..2}\left(\frac{1}{i},\frac{1}{i-2},..\frac{1}{3},1\right)^2a_0 \text{ when } i \text{ is even.}$$

And from the way in which the coefficients are formed, it is easy to see that in the case of $\nabla_1^m u$,

$$ia_i=\left(\frac{1}{i}\right)^{m-1}A_{i-1}$$

$$-\frac{n-i+2}{i-2}\left(\frac{1}{i},\frac{1}{i-2}\right)^mA_{i-3}$$

$$+\frac{(n-i+2)(n-i+4)}{(i-2)(i-4)}\left(\frac{1}{i},\frac{1}{i-2},\frac{1}{i-4}\right)^mA_{i-5}$$

$$-\ldots$$

with conditions for i even and i odd, similar to those given in the cases of ∇_1^2, ∇_1^3,

And since it has been shown above that

$$F(\nabla_1)u=\Sigma\left\{F\left((n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right)\right\}a_ix^{n-i}y$$

it follows that, if

$$\left\{F(\nabla_1)\right\}^{-1}U=u,$$

the coefficients of u will be given by the equation

$$a_i=\left\{F\left((n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right)\right\}^{-1}A_i.$$

Still more generally,

$$F(\nabla, \nabla_1)u=\Sigma\left\{F\left(n, (n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}\right)\right\}a_i x^{n-i}y.$$

So that the effect of the operation

$$(a\,b\,..\,\between\Xi\,\Xi_1)^m,$$

or, as it may also be written,

$$(a\,b\,..\,\between x\,y)^m\frac{d^m}{dx^m}+\frac{n}{1}(a\,b\,..\,\between xy^{m-1}\between yx)\frac{d^m}{dx^{m-1}dy}+..,$$

will be exhibited by replacing ∇, ∇_1 by n, and $(n-i+1)\varepsilon^{-\frac{d}{di}}+(i+1)\varepsilon^{\frac{d}{di}}$ respectively in either of the values given for $(ab\,..\,\between\Xi\,\Xi_1)^m$ in pages 16 or 19.

In order, therefore, to solve the differential equation

$$(a\,b\,..\,\between\Xi\,\Xi_1)^m u=0,$$

we must first reduce it to the form

$$F(\nabla\,\nabla_1)u=0;$$

then solving the symbolical equation

$$K^{-1}F(\nabla\,\nabla_1)=0$$

(where K is the coefficient of the highest power of ∇_1) with respect to ∇_1, and calling the roots $f_1(\nabla), f_2(\nabla), ..f_m(\nabla)$, or simply $f_1, f_2, ..f_m$, we have

$$u=\frac{0}{F(\nabla, \nabla_1)}=\frac{0}{K(\nabla_1-f_1)(\nabla_1-f_2)\,..\,(\nabla_1-f_m)}$$

$$=\frac{1}{K}\left(\frac{C_1}{\nabla_1-f_1}+\frac{C_2}{\nabla_1-f_2}+\,..\,\frac{C_m}{\nabla_1-f_m}\right)0,$$

where

$$C_1=\frac{f_1^{m-1}}{(f_1-f_2)(f_1-f_3)\,..\,(f_1-f_m)},\quad C_2=\frac{f_2^{m-1}}{(f_2-f_1)(f_2-f_3)\,..\,(f_2-f_m)},\,..$$

In order to evaluate the expression for u, let $p_1, p_2, ..p_m$ be values of ∇ which make $f_1, f_2, ..f_m$ vanish. They may in fact be called roots of $f_1, f_2, ..f_m$; but as these functions are generally irrational, they cannot be replaced by the products of factors of the form $\nabla-p$. In general there will be only one quantity p for each function f; because if f be rationalized, it will give rise to a function of the degree m; but although the equation so formed will in general have m roots, $(m-1)$ of them will in fact be extraneous to the particular equation rationalized, and belong one apiece to each of the remaining equations of the system.

Let, then, $u_{p_1}, u_{p_2}, \ldots u_{p_m}$ be arbitrary homogeneous functions of x, y, of the degrees p_1, $p_2, \ldots p_m$ respectively; then, since

$$f(\nabla)u_p = f(p)u_p = 0, \text{ because } f(p)=0,$$

we may make

$$\frac{0}{f(\nabla)} = u_p,$$

and consequently

$$u = \frac{1}{K}\Sigma\frac{C_i}{\nabla_1 - f_i}0 = -\frac{1}{K}\Sigma\frac{C_i}{1-\frac{\nabla_1}{f_i}}\cdot\frac{0}{f_i} = -\frac{1}{K}\Sigma\frac{C_i}{1-\frac{\nabla_1}{f_i}}u$$

or replacing C_i by its value,

$$u = -\frac{1}{K}\Sigma\frac{f_i^m}{(f_i-f_1)(f_i-f_2)}\cdots\frac{1}{f_i-\nabla_1}u_{p_i}$$

$$= \frac{1}{K}\Sigma\frac{f_i(p_i)m}{\big(f_i(p_i)-f_1(p_i)\big)\big(f_i(p_i)-f_2(p_i)\big)}\cdots\frac{1}{\nabla_1 - f_i(p_i)}u_{p_i}.$$

This expression, however, leads only to illusory results, for $f_i(p_i)=0$; and consequently on developing $\frac{1}{\nabla_1 - f_i(p_i)}$, even in descending powers of $f_i(p_i)$, the first m terms vanish, and all the terms after the $(m+1)$th become infinite; the $(m+1)$th alone being finite.

The following method is, however, free from this difficulty. Select any second suffix j; then

$$u = -\frac{1}{K}\Sigma\frac{C_i}{f_i - \nabla_1}0$$

$$= -\frac{1}{K}\Sigma\left\{\frac{f_i^{m-1}}{(f_i-f_1)\ldots(f_i-f_j)\ldots(f_i-\nabla_1)\ldots(f_i-f_m)} + \frac{f_j^{m-1}}{(f_j-f_1)\ldots(f_j-\nabla_1)\ldots(f_j-f_i)\ldots(f_j-f_m)}\right\}0$$

$$= -\frac{1}{K}\Sigma\left\{\frac{f_i^{m-1}}{(f_i-f_1)\ldots(f_i-\nabla_1)\ldots(f_i-f_m)} - \frac{f_j^{m-1}}{(f_j-f_1)\ldots(f_j-\nabla_1)\ldots(f_j-f_m)}\right\}\frac{0}{f_i-f_j}.$$

And if α_{ij}, $\beta_{ij}, \ldots$ be the roots of

$$f_i - f_j = 0$$

when solved with respect to ∇, we shall have

$$\frac{0}{f_i-f_j} = A_{ij}u_{\alpha_{ij}} + B_{ij}u_{\beta_{ij}} + \ldots$$

and

$$u = -\frac{1}{K}\Sigma\Bigg\{\Bigg[\frac{f_i(\alpha_{ij})^{m-1}}{\big(f_i(\alpha_{ij})-f_1(\alpha_{ij})\big)\ldots\big(f_i(\alpha_{ij})-\nabla_1\big)\ldots\big(f_i(\alpha_{ij})-f_m(\alpha_{ij})\big)}$$

$$-\frac{f_j(\alpha_{ij})^{m-1}}{\big(f_j(\alpha_{ij})-f_1(\alpha_{ij})\big)\ldots\big(f_j(\alpha_{ij})-\nabla_1\big)\ldots\big(f_j(\alpha_{ij})-f_m(\alpha_{ij})\big)}\Bigg]A_{ij}u_{\alpha_{ij}}$$

$$+ \text{ similar terms in } B_{ij},\ \beta_{ij};\ \ldots\Bigg\}.$$

And if we assume

$$u_\alpha=(a_0\, a_1 \,..\,\between x\, y)^\alpha$$
$$u_\beta=(b_0\, b_1 \,..\,\between x\, y)^\beta$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

the values of the coefficients, as affected by the operative factor, may be calculated by formulæ given above.

§ 2.

Let s, s_1, .. be any linear functions of the variables x, y, such as

$$s_i=\alpha_i x+\beta_i y,$$

and let

$$s_i\frac{d}{dx}=u_i,\qquad s_i\frac{d}{dy}=v_i.$$

It is proposed in the first place to investigate expressions for symbols of the form

$$s_1\, s_2 \,..\, s_i\frac{d^i}{dx^{i-j}\,dy^j}$$

in terms of the us and vs.

First, as is easily seen,

$$ss_1\frac{d^2}{dx^2}=s\frac{d}{dx}\left(s_1\frac{d}{dx}-\alpha_1\right)=u(u_1-\alpha_1),$$

$$ss_1\frac{d^2}{dx\,dy}=s\frac{d}{dx}\cdot s_1\frac{d}{dy}-s\alpha_1\frac{d}{dy}=s\frac{d}{dy}\cdot s_1\frac{d}{dx}-s\beta_1\frac{d}{dx};$$

therefore

$$2ss_1\frac{d^2}{dx\,dy}=s\frac{d}{dx}\left(s_1\frac{d}{dy}-\beta_1\right)+s\frac{d}{dy}\left(s_1\frac{d}{dx}-\alpha_1\right)$$
$$=u(v_1-\beta_1)+v(u_1-\alpha_1).$$

Hence the quadratic system,

$$s_{00}\,s_{01}\frac{d^2}{dx^2}=u_{00}(u_{01}-\alpha_{01}),$$

$$2s_{10}s_{11}\frac{d^2}{dx\,dy}=u_{10}(v_{11}-\beta_{11})+v_{10}(u_{11}-\alpha_{11}),$$

$$s_{20}\,s_{21}\frac{d^2}{dy^2}=v_{20}(v_{21}-\beta_{21}),$$

or

$$(a_0x^2+2b_0xy+c_0y^2)\frac{d^2}{dx^2}+2(a_1x^2+2b_1\,xy+c_1y^2)\frac{d^2}{dx\,dy}+(a_2x^2+2b_2xy+c_2y^2)\frac{d^2}{dy^2}$$
$$=u_{00}u_{01}+u_{10}v_{11}+v_{10}u_{11}+v_{20}v_{21}-\alpha_{01}u_{00}-\beta_{11}u_{10}-\alpha_{11}v_{10}-\beta_{21}v_{20}.$$

Again, developing the following expression by EULER'S theorem, and remembering that, since the ss are linear, their second and higher differential coefficients vanish, we have

$$ss_1\frac{d^2}{dx^2}\cdot s_2\frac{d}{dx}=ss_1s_2\frac{d^3}{dx^3}+2ss_1\alpha_2\frac{d^2}{dx^2},$$

or

$$\begin{aligned}ss_1s_2\frac{d^3}{dx^3}&=ss_1\frac{d^2}{dx^2}\left(s_2\frac{d}{dx}-2c_2\right)\\&=s\frac{d}{dx}\left(s_1\frac{d}{dx}-\alpha_1\right)\left(s_2\frac{d}{dx}-2\alpha_2\right)\\&=u(u_1-\alpha_1)(u_2-2\alpha_2).\end{aligned}$$

Again, taking the first two equations of the quadratic system above written, and operating on $s_2\frac{d}{dy}$, $s_2\frac{d}{dx}$ respectively, we have

$$ss_1\frac{d^2}{dx^2}\cdot s_2\frac{d}{dy}=ss_1s_2\frac{d^3}{dx^2\,dy}+2ss_1\alpha_2\frac{d^2}{dx\,dy},$$

$$2ss_1\frac{d^2}{dx\,dy}\cdot s_2\frac{d}{dx}=2ss_1s_2\frac{d^3}{dx^2\,dy}+2ss_1\alpha_2\frac{d^2}{dx\,dy}+2ss_1\beta_2\frac{d^2}{dx^2}.$$

Hence

$$\begin{aligned}3ss_1s_2\frac{d^3}{dx^2\,dy}&=ss_1\frac{d^2}{dx^2}\left(s_2\frac{d}{dy}-2\beta_2\right)+2ss_1\frac{d^2}{dx\,dy}(s_2-2\alpha_2)\\&=u(u_1-\alpha_1)(v_2-2\beta_2)+\{u(v_1-\beta_1)+v(u_1-\alpha_1)\}(u_2-2\alpha_2)\\&=u(u_1-\alpha_1)(v_2-2\beta_2)+u(v_1-\beta_1)(u_2-2\alpha_2)+v(u_1-\alpha_1)(u_2-2\alpha_2).\end{aligned}$$

Hence the cubic system,

$$ss_1s_2\frac{d^3}{dx^3}=u(u_1-\alpha_1)(u_2-2\alpha_2),$$

$$\begin{aligned}3ss_1s_2\frac{d^3}{dx^2\,dy}=&\,u(u_1-\alpha_1)(v_2-2\beta_2)\\&+u(v_1-\beta_1)(u_2-2\alpha_2)\\&+v(u_1-\alpha_1)(u_2-2\alpha_2),\end{aligned}$$

$$\begin{aligned}3ss_1s_2\frac{d^3}{dx\,dy^2}=&\,v(v_1-\beta_1)(u_2-2\alpha_2)\\&+v(u_1-\alpha_1)(v_2-2\beta_2)\\&+u(v_1-\beta_1)(v_2-2\beta_2),\end{aligned}$$

$$ss_1s_2\frac{d^3}{dy^3}=v(v_1-\beta_1)(v_2-2\beta_2),$$

in which the ss, us, vs, αs, βs may be furnished with double suffixes, as was done in the quadratic system; and then the sum of the four equations would give the value of

$$(abcd\between xy)^3\frac{d^3}{dx^3}+(a_1b_1c_1d_1\between xy)^3\frac{d^3}{dx^2\,dy}+\,..$$

For the general case, let

$$N_p ss_1\,..\,s_{i-1}\frac{d^i}{dx^{i-p}\,dy^p}=A,$$

$$N_{p+1}ss_1\,..\,s_{i-1}\frac{d^i}{dx^{i-p-1}\,dy^{p+1}}=B,$$

where

$$\mathrm{N}_p=\frac{i(i-1)\,..\,(i-p+1)}{1\,.\,2\,..\,p},\quad \mathrm{N}_{p+1}=\frac{i(i-1)\,..\,(i-p)}{1\,.\,2\,..\,(p+1)};$$

so that A and B are two consecutive terms in the system of the degree i. Then operating upon $s_i\frac{d}{dy}$ with A, and upon $s_i\frac{d}{dx}$ with B, in the same way as was done in the case of the third degree, we have

$$\begin{aligned}\mathrm{A}s_i\frac{d}{dy}&=\mathrm{N}_p s\,s_1\,..\,s_i\frac{d^{i+1}}{dx^{i-p}dy^{p+1}}\\&+\mathrm{N}_p s s_1\,..\,s_{i-1}(i-p)\alpha_i\frac{d^i}{dx^{i-p-1}dy^{p+1}}\\&+\mathrm{N}_p s s_1\,..\,s_{i-1}p\beta_i\frac{d^i}{dx^{i-p}dy^p},\end{aligned}$$

$$\begin{aligned}\mathrm{B}s_1\frac{d}{dx}&=\mathrm{N}_{p+1}s s_1\,..\,s_i\frac{d^{i+1}}{dx^{i-p}dy^{p+1}}\\&+\mathrm{N}_{p+1}s s_1\,..\,s_{i-1}(i-p-1)\alpha_i\frac{d^i}{dx^{i-p-1}dy^{p+1}}\\&+\mathrm{N}_{p+1}s s_1\,..\,s_{i-1}(p+1)\beta_i\frac{d^i}{dx^{i-p}dy^p}.\end{aligned}$$

But

$$(p+1)\mathrm{N}_{p+1}=(i-p)\mathrm{N}_p$$

and

$$\begin{aligned}\mathrm{N}_p+\mathrm{N}_{p+1}&=\frac{i(i-1)\,..\,(i-p+1)}{1\,.\,2\,..\,(p+1)}(p+1+i-p)\\&=\frac{(i+1)i\,..\,(i-p+1)}{1\,.\,2\,..\,(p+1)};\end{aligned}$$

in other words, $\mathrm{N}_p+\mathrm{N}_{p+1}$ is equal to the $(p+1)$th coefficient in the case of the degree $(i+1)$. Hence, adding the above written expressions for $\mathrm{A}s_i\frac{d}{dy}$, $\mathrm{B}s_i\frac{d}{dx}$, and calling the value of $\mathrm{N}_p+\mathrm{N}_{p+1}$ ${}_{i+1}\mathrm{N}_{p+1}$, we have

$$\begin{aligned}&{}_{i+1}\mathrm{N}_{p+1}s s_1\,..\,s_i\frac{d^{i+1}}{dx^{i-p}dy^{p+1}}\\&=\mathrm{A}\Big(s_i\frac{d}{dy}-i\beta_i\Big)+\mathrm{B}\Big(s_i\frac{d}{dx}-i\alpha_i\Big).\end{aligned}$$

But if it is true (and it has been proved in the cases of 2, 3, . .) that

$$\mathrm{A}=\mathrm{S}u(u_1-\alpha_1)\,..\,\big(u_{i-p}-(i-p)\alpha_{i-p}\big)\big(v_{i-p+1}-(i-p+1)\beta_{i-p+1}\big)\,..\,\big(v_{i-1}-(i-1)\beta_{i-1}\big),$$
$$\mathrm{B}=\mathrm{S}u(u_1-\alpha_1)\,..\,\big(u_{i-p-1}-(i-p-1)\alpha_{i-p-1}\big)\big(v_{i-p}-(i-p)\beta_{i-p}\big)\,..\,\big(v_{i-1}-(i-1)\beta_{i-1}\big);$$

i. e. if A and B are respectively equal to the sums of all the products of the above forms that can be formed by interchanging the *u*s and *v*s, so that the total number of the *u*s and of the *v*s remains constant in each product (viz. $(i-p+1)$ *u*-factors and $(p-1)$ *v*-factors in A, and $(i-p)$ *u*-factors and p *v*-factors in B); then will $\mathrm{A}(v_i-i\beta_i)+\mathrm{B}(u_i-i\alpha_i)$

be equal to the sum of all the products of the above form which can be formed with $(i-p)$ u-factors and $(p+1)$ v-factors.

In order to calculate the effect of the operation $\mathrm{S}u(u_1-\alpha_1)\ldots$ upon a given function, let

$$\mathrm{V}=\Sigma a_i x^{n-i}y^i,$$

then

$$\begin{aligned} u\mathrm{V} &= \Sigma(\alpha x+\beta y)(n-i)a_i x^{n-i-1}y^i \\ &= \Sigma\left\{(n-i)\alpha a_i x^{n-i}y^i+(n-i)\beta a_i x^{n-i-1}y^{i+1}\right\} \\ &= \Sigma\left\{(n-i)\alpha a_i+(n-i+1)\beta a_{i-1}\right\}x^{n-i}y^i \\ &= \Sigma\left\{(\alpha+\beta\varepsilon^{-\frac{d}{di}})(n-i)a_i\right\}x^{n-i}y^i, \end{aligned}$$

in which $\alpha+\beta\varepsilon^{-\frac{d}{di}}$ must be considered as an operative factor affecting the quantities within the brackets { } alone.

Similarly,

$$(u_1-\alpha_1)\mathrm{V}=\Sigma\left\{(n-i-1)\alpha_1 a_i+(n-i+1)\beta_1 a_{i-1}\right\}x^{n-i}y^i;$$

therefore

$$\begin{aligned} u(u_1-\alpha_1)\mathrm{V} &= \Sigma\left\{(\alpha+\beta^{-\frac{d}{di}})(n-i)(n-i-1)\alpha_1 a_i+(\alpha+\beta\varepsilon^{-\frac{d}{di}})(n-i)(n-i+1)\beta_1 a_{i-1}\right\}x^{n-i}y^i \\ &= \Sigma\left\{(\alpha+\beta\varepsilon^{-\frac{d}{di}})(\alpha_1+\beta_1\varepsilon^{-\frac{d}{di}})(n-i)(n-i-1)a_i\right\}x^{n-i}y^i. \end{aligned}$$

And if, as has been proved in the cases 1, 2, . . ,

$$u(u_1-\alpha_1)\ldots(u_m-m\alpha_m)\mathrm{V}$$

$$=\Sigma\left\{(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_m+\beta_m\varepsilon^{-\frac{\partial}{di}}(n-i)(n-i-1)\ldots(n-i-m)a_i\right\}x^{n-i}y^i,$$

then

$$\left(u_{m+1}-(m+1)\alpha_{m+1}\right)\mathrm{V}=\Sigma\left\{(n-i-m-1)\alpha_{m+1}a_i+(n-i+1)\beta_{m+1}a_{i-1}\right\}x^{n-i}y^i;$$

and generally,

$$u(u_1-\alpha_1)\ldots\left(u_{m+1}-(m+1)\alpha_{m+}\right)\mathrm{V}$$

$$\begin{aligned} &=\Sigma\left\{(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_m+\beta_m\varepsilon^{-\frac{\partial}{di}})(n-i)(n-i-1)\ldots(n-i-m-1)\alpha_{m+1}a_i\right. \\ &\quad\left.+(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_m+\beta_m\varepsilon^{-\frac{\partial}{di}})(n-i+1)(n-i)\ldots(n-i-m)\beta_{m+1}a_{i-1}\right\}x^{n-i}y^i \\ &=\Sigma\left\{(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_{m+1}+\beta_{m+1}\varepsilon^{-\frac{\partial}{di}})(n-i)(n-i-1)\ldots(n-i-m-1)a_i\right\}x^{n-i}y^i. \end{aligned}$$

Again,

$$\begin{aligned} v\mathrm{V} &= \Sigma\left\{(\alpha x+\beta y)ia_i\right\}x^{n-i}y^i \\ &= \Sigma\left\{\alpha(i+1)a_{i+1}+\beta ia_i\right\}x^{n-i}y^i \\ &= \Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)ia_i\right\}x^{n-i}y^i \\ (v_1-\beta_1)\mathrm{V} &= \Sigma\left\{\alpha_1(i+1)a_{i+1}+\beta_1(i-1)a_i\right\}x^{n-i}y^i \\ v(v_1-\beta_1)\mathrm{V} &= \Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)i[\alpha_1(i+1)a_{i+1}+\beta_1(i-1)a_i]\right\}x^{n-i}y^i \\ &= \Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1\varepsilon^{\frac{\partial}{di}}+\beta_1)i(i-1)a_i\right\}x^{n-i}y^i. \end{aligned}$$

And in the same way as in the case of the us, it may be shown that

$$v(v_1-\beta_1)\,..\,(v_m-m\beta_m)\mathrm{V}$$
$$=\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1\varepsilon^{\frac{\partial}{di}}+\beta_1)\,..\,(\alpha_m+\beta_m\varepsilon^{\frac{\partial}{di}})i(i-1)\,..\,(i-m)a_i\right\}x^{n-i}y^i.$$

Again, for the intermediate terms of the system,

$$u(v_1-\beta_1)\mathrm{V}=\Sigma\left\{(\alpha+\beta\varepsilon^{\frac{\partial}{di}})(n-i)[(i+1)\alpha_1 a_{i+1}+(i-1)\beta_1\alpha_i]\right\}x^{n-i}y^i$$
$$=\Sigma\left\{(\alpha^{\frac{\partial}{di}}+\beta)(n-i+1)[i\alpha_1 a_i+(i-2)\beta_1 a_{i-1}]\right\}x^{n-i}y^i$$
$$v(u_1-\alpha_1)\ =\Sigma\left\{(\alpha^{\frac{\partial}{di}}+\beta)i[(n-i-1)\alpha_1 a_i+(n-i+1)\beta_1 a_{i-1}]\right\}x^{n-i}y^i.$$

Hence

$$\left\{u(v_1-\beta_1)+v(u_1-\alpha_1)\right\}\mathrm{V}$$
$$=2\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)[(n-i)i\alpha_1 a_i+(n-i+1)(i-1)\beta_1 a_{i-1}]\right\}x^{n-i}y^i$$
$$=2\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})(n-i)ia_i\right\}x^{n-i}y^i.$$

Again,

$$\left\{u(v_1-\beta_1)+v(u_1-\alpha_1)\right\}(u_2-2\alpha_2)\mathrm{V}$$
$$=2\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})(n-i)i[(n-i-2)\alpha_2 a_i+(n-i+1)\beta_2 a_{i-1}]\right\}x^{n-i}y^i,$$
$$u(u_1-\alpha_1)(v_2-2\beta_2)\mathrm{V}$$
$$=\ \Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})(n-i+1)(n-i)[\alpha_2 ia_i+\beta_2(i-3)a_{i-1}]\right\}x^{n-i}y^i.$$

Hence

$$u(u_1-\alpha_1)(v_2-2\beta_2)+u(v_1-\beta\ )(u_2-2\alpha_2)+v(u_1-\alpha_1)(u_2-2\alpha_2)$$
$$=3\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})[(n-i)i\alpha_2 a_2(n-i-1)+(n-i+1)(n-i)\beta_2 a_{i-1}(i-1)]\right\}x^{n-i}y^i$$
$$=3\Sigma\left\{(\alpha\varepsilon^{\frac{\partial}{di}}+\beta)(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})(\alpha_2+\beta_2\varepsilon^{-\frac{\partial}{di}})(n-i)(n-i-1)ia_i\right\}x^{n-i}y^i.$$

Again, for the general term, let

$$\mathrm{A}={}_m\mathrm{N}_{p-1}ss_1\,..\,s_{m-1}\frac{d^m}{dx^{m-p+1}dy^{p-1}}$$
$$={}_m\mathrm{N}_{p-1}(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})\,..\,(\alpha_{m-p}+\beta_{m-p}\varepsilon^{-\frac{\partial}{di}})(\alpha_{m-p+1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-p+1})\,..\,(\alpha_{m-1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-1})(n-i)\,..\,(n-i-m+p)i\,..\,(i-p+2),$$
$$\mathrm{B}={}_m\mathrm{N}_{p-1}ss_1\,..\,s_{m-1}\frac{d^m}{dx^{m-p}dy^p}$$
$$={}_m\mathrm{N}_{p-1}(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})\,..\,(\alpha_{m-p-1}+\beta_{m-p-1}\varepsilon^{-\frac{\partial}{di}})(\alpha_{m-p}\varepsilon^{\frac{\partial}{di}}+\beta_{m-p})\,..\,(\alpha_{m-1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-1})(n-i)\,..\,(n-i-m+p-1)i\,..\,(i-p+1).$$

Also let

$$\mathrm{K}=(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})\,..\,(\alpha_{m-p}+\beta_{m-p}\varepsilon^{-\frac{\partial}{di}})(\alpha_{m-p+1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-p+1})\,..\,(\alpha_{m-1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-1}).$$

Then changing $\alpha_{m-p}\varepsilon^{\frac{\partial}{di}}+\beta_{m-p}$, in B, into $\alpha_{m-p}+\beta_{m-p}\varepsilon^{\frac{\partial}{di}}$, and compensating the change by writing $(i+1)$ for i in the numerical factors to the right, we have

$$\begin{aligned}
A&=K\ {}_m N_{p-1}(n-i)\ldots(n-i-m+p)i\ldots(i-p+2),\\
B&=K\ {}_m N_p(n-i-1)\ldots(n-i-m+p)(i+1)\ldots(i-p+2)\varepsilon^{\frac{\partial}{di}},
\end{aligned}$$

the factor $\varepsilon^{\frac{\partial}{di}}$ having been added in order to indicate that i must be changed into $(i+1)$ in the expression upon which B will presently be made to operate. Then making

$$\begin{aligned}
H&=(n-i-1)\ldots(n-i-m+p)i\ldots(i-p+2),\\
C&=\left\{A(v_m-m\beta_m)+B(u_m-m\alpha_m)\right\}V\\
&=\Sigma\Big[KH\ {}_m N_{p-1}(n-i)\left\{(i+1)\alpha_m a_{i+1}+(i-m)\beta_m a_i\right\}\\
&\qquad+KH\ {}_m N_p\ \ (i+1)\left\{(n-i-m-1)\alpha_m a_{i+1}+(n-i)\beta_m a_i\right\}\Big]x^{n-i}y^i\\
&=\Sigma\Big[\!\!\Big[KH\left\{(i+1)[(n-i)\,{}_m N_{p-1}+(n-i-m-1)\,{}_m N_p]\alpha_m a_{i+1}\right.\\
&\qquad\left.+(n-i)[(i-m)\,{}_m N_{p-1}+(i+1)\ \ {}_m N_p]\beta_m a_i\right\}\Big]\!\!\Big].
\end{aligned}$$

Now

$$\begin{aligned}
&(n-i)\,{}_m N_{p-1}+(n-i-m-1)\,{}_m N_p\\
&=(n-i)({}_m N_{p-1}+{}_m N_p)-(m+1)\,{}_m N_p\\
&=(n-i)\,{}_{m+1}N_p-(m-p+1)\,{}_{m+1}N_p\\
&=(n-i-m+p-1)\,{}_{m+1}N_p,
\end{aligned}$$

and

$$\begin{aligned}
&(i-m)\,{}_m N_{p-1}+(i+1)\,{}_m N_p\\
&=(i+1)({}_m N_{p-1}+{}_m N_p)-(m+1)\,{}_m N_{p-1}\\
&=(i+1)\,{}_{m+1}N_p-p\,{}_{m+1}N_p\\
&=(i+1-p)\,{}_{m+1}N_p.
\end{aligned}$$

Hence

$$\begin{aligned}
C&=\Sigma\Big[\!\!\Big[{}_{m+1}N_p KH\left\{(n-i-m+p-1)(i+1)\alpha_m a_{i+1}+(n-i)(i-p+1)\beta_m a_i\right\}\Big]\!\!\Big]x^{n-i}y^i\\
&=\Sigma\Big[\!\!\Big[{}_{m+1}N_p K(\alpha_m\varepsilon^{\frac{\partial}{di}}+\beta_m)(n-i)\ldots(n-i-m+p)i\ldots(i-p+1)a_i\Big]\!\!\Big]x^{n-i}y^i\\
&=\Sigma\Big[\!\!\Big[{}_{m+1}N_p(\alpha+\beta\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_{m-p}+\beta_{m-p}\varepsilon^{-\frac{\partial}{di}})(\alpha_{m-p+1}\varepsilon^{\frac{\partial}{di}}+\beta_{m-p+1})\ldots(\alpha_m\varepsilon^{\frac{\partial}{di}}+\beta_m)\\
&\qquad(n-i)\ldots(n-i-m+p)i\ldots(i-p+1)a_i\Big]\!\!\Big]x^{n-i}y^i,
\end{aligned}$$

which proves the formula generally.

This expression may, however, be transformed into a more convenient shape, as follows. Let

$$s_1\,s_2\ldots s_m=(\mathrm{a}\,\mathrm{a}_1\ldots\mathrm{a}_m\between x\,y)^m,$$

then

$$(\alpha_1+\beta_1\varepsilon^{-\frac{\partial}{di}})(\alpha_2+\beta_2\varepsilon^{-\frac{\partial}{di}})\ldots(\alpha_m+\beta_m\varepsilon^{-\frac{\partial}{di}})=(\mathrm{a}\,\mathrm{a}_1\ldots\mathrm{a}_m\between 1\ \varepsilon^{-\frac{\partial}{di}})^m.$$

And if

$$\mathrm{U}=(\mathrm{a}\,\mathrm{a}_1\,..\,\mathrm{a}_m)(x\,y)^m,$$

$$\Upsilon=(\mathrm{a}\,\mathrm{a}_1\,..\,\mathrm{a}_m)(\varepsilon^{\frac{d}{di}}1)^m,$$

$$\mathrm{F}=(a\,a_1\,..\,a_m)(x\,y)^n,$$

$$\mathrm{P}_i={}_n\mathrm{N}_i i(i-1)\,..\,(i-m+1),$$

then, since

$$(n-i+m)(n-i+m-1)\,..\,(n-i+p+1)(i-m)\,..\,(i-m-p+1)\varepsilon^{-m\frac{d}{di}}\,{}_n\mathrm{N}_i$$

$$=(n-i+m)(n-i+m-1)\,..\,(n-i+p+1)(i-m)\,..\,(i-m-p+1)\,{}_n\mathrm{N}_{i-m}\varepsilon^{-\frac{d}{di}}$$

$$={}_n\mathrm{N}_{i-p}(i-m+1)(i-m+2)\,..\,(i-p)(i-m)(i-m-1)\,..\,(i-m-p+1)\varepsilon^{-\frac{d}{di}}$$

$$={}_n\mathrm{N}_{i-p}(i-p)(i-p-1)\,..\,(i-m-p+1)\varepsilon^{-m\frac{d}{di}},$$

we have

$$\mathrm{U}\frac{d^m\mathrm{F}}{dx^{m-p}\,dy^p}=\Sigma\left\{\Upsilon\varepsilon^{-(m-p)\frac{d}{di}}(n-i)\,..\,(n-i-m+p+1)i\,..\,(i-p+1)\,{}_n\mathrm{N}_i a_i\right\}x^{n-i}y^i$$

$$=\Sigma\left\{\Upsilon\varepsilon^{p\frac{d}{di}}(n-i+m)\,..\,(n-i+p+1)(i-m)\,..\,(i-m-p+1)\varepsilon^{-m\frac{d}{di}}\,{}_n\mathrm{N}_i a_i\right\}x^{n-i}y^i$$

$$=\Sigma\left\{\Upsilon\varepsilon^{p\frac{d}{di}}\mathrm{P}_{i-p}\varepsilon^{-m\frac{d}{di}}a_i\right\}x^{n-i}y^i$$

$$=\Sigma\left\{\Upsilon\mathrm{P}_i\varepsilon^{p\frac{d}{di}}\,a_{i-m}\right\}x^{n-i}y^i.$$

Hence

$$\mathrm{U}_0\frac{d^m\mathrm{F}}{dx^m}+\frac{m}{1}\mathrm{U}_1\frac{d^m\mathrm{F}}{dx^{m-1}dy}+\,..\,+\mathrm{U}_m\frac{d^m\mathrm{F}}{dy^m}$$

$$=\Sigma\left\{\left[\Upsilon_0\mathrm{P}_i+\frac{m}{1}\Upsilon_1\mathrm{P}_i\varepsilon^{\frac{d}{di}}+\,..\,+\Upsilon_m\mathrm{P}_i\varepsilon^{m\frac{d}{di}}\right]a_{i-m}\right\}x^{n-i}y^i$$

$$=\Sigma\left\{\left[\left(\mathrm{a}_{00}\varepsilon^{m\frac{d}{di}}+\frac{m}{1}\mathrm{a}_{01}\varepsilon^{(m-1)\frac{d}{di}}+\,..\,+\mathrm{a}_{0m}\right)\mathrm{P}_i\right.\right.$$

$$+\frac{m}{1}\left(\mathrm{a}_{10}\varepsilon^{m\frac{d}{di}}+\frac{m}{1}\mathrm{a}_{11}\varepsilon^{(m-1)\frac{d}{di}}+\,..\,+\mathrm{a}_{1m}\right)\mathrm{P}_i\varepsilon^{\frac{d}{di}}$$

$$+\,\ldots$$

$$\left.\left.+\left(\mathrm{a}_{m0}\varepsilon^{m\frac{d}{di}}+\frac{m}{1}\mathrm{a}_{m1}\varepsilon^{(m-1)\frac{d}{di}}+\,..\,+\mathrm{a}_{mm}\right)\mathrm{P}_i\varepsilon^{m\frac{d}{di}}\right]a_{i-m}\right\}x^{n-i}y^i;$$

and writing

$$\frac{m(m-1)\,..\,(m-q+1)}{1\,.\,2\,..\,q}=\mathrm{M}_q,$$

the above expression

$$=\Sigma\left\{\left[\mathrm{M}_0\mathrm{a}_{0,\,m}\mathrm{P}_i\right.\right.$$

$$+(\mathrm{M}_1\mathrm{a}_{0,\,m-1}\mathrm{P}_{i+1}+\mathrm{M}_1\mathrm{a}_{1,\,m}\mathrm{P}_i)\varepsilon^{\frac{d}{di}}$$

$$+(\mathrm{M}_0\mathrm{M}_2\mathrm{a}_{0,\,m-2}\mathrm{P}_{i+2}+\mathrm{M}_1^2\mathrm{a}_{1,\,m-1}\mathrm{P}_{i+1}+\mathrm{M}_2\mathrm{M}_0\mathrm{a}_{2,\,m}\mathrm{P}_i)\varepsilon^{2\frac{d}{di}}$$

$$+\,\ldots$$

$$+(\mathrm{M}_1\mathrm{a}_{m-1,\,0}\mathrm{P}_{i+m}+\mathrm{M}_1\mathrm{a}_{m,\,1}\mathrm{P}_{i+m-1})\varepsilon^{(2m-1)\frac{d}{di}}$$

$$\left.\left.+\ \mathrm{M}_0\mathrm{a}_{m,\,0}\mathrm{P}_{i+m}\varepsilon^{2m\frac{d}{di}}\right]a_{i-m}\right\}x^{n-i}y^i;$$

in which all terms will vanish for which

$$i-m<0, \text{ and } i+m>n.$$

In order to determine the effect of the operation

$$\left[\left(U_0 U_1 \ldots \between \frac{d}{dx}\frac{d}{dy}\right)^m\right]^{-1}$$

on a given function, we may proceed, as before, by making

$$\left[\left(U_0 U_1 \ldots \between \frac{d}{dx}\frac{d}{dy}\right)^m\right]^{-1}(A_0 A_1 \ldots \between xy)^n = (a_0 a_1 \ldots \between xy)^n,$$

or

$$(A_0 A_1 \ldots \between xy)^n = \left(U_0 U_1 \ldots \between \frac{d}{dx}\frac{d}{dy}\right)^m (a_0 a_1 \ldots \between xy)^n;$$

and then the coefficients $a_0 a_1 \ldots$ will be given by the system, of which the following is a type:

$$\begin{aligned} A_i = [&M_0 \mathrm{a}_{0,m} P_i \\ &+(M_1 \mathrm{a}_{0,m-1} P_{i+1} + M_1 \mathrm{a}_{1m} P_i)\varepsilon^{\frac{d}{di}} \\ &+ \ldots]a_{i-m}, \end{aligned}$$

as above. In order to solve these equations for $a\, a_1 \ldots$, we may regard the expression within the brackets [ ] as a function of $\varepsilon^{\frac{d}{di}}$; and resolving it into its factors, we may proceed by way of operations instead of direct elimination. Let $p_1, p_2, \ldots p_{2m}$ be the roots; then

$$A_i = \mathrm{a}_{m,0} P_{i+m}(\varepsilon^{\frac{d}{di}} - p_1)(\varepsilon^{\frac{d}{di}} - p_2) \ldots (\varepsilon^{\frac{d}{di}} - p_{2m})a_{i-m}$$

and

$$a_{i-m} = (\varepsilon^{\frac{d}{di}} - p_1)^{-1}(\varepsilon^{\frac{d}{di}} - p_2)^{-1} \ldots (\varepsilon^{\frac{d}{di}} - p_{2m})^{-1}\frac{A_i}{\mathrm{a}_{m,0} P_{i+m}}.$$

But

$$\begin{aligned} (\varepsilon^{\frac{d}{di}} - p_1)^{-1}\frac{A_i}{P_{i+m}} &= -\frac{1}{p_1}\left(1 + \frac{\epsilon^{\frac{d}{di}}}{p_1} + \frac{\epsilon^{2\frac{d}{di}}}{p_1^2} + \ldots\right)\frac{A_i}{P_{i+m}} \\ &= -\frac{1}{p_1}\left(\frac{A_i}{P_{i+m}} + \frac{1}{p_1}\frac{A_{i+1}}{P_{i+m+1}} + \ldots + \frac{A_n}{p_1^{n-i}P_{n+m+1}}\right) \end{aligned}$$

$$(\varepsilon^{\frac{d}{di}} - p_1)^{-1}(\varepsilon^{\frac{d}{di}} - p_2)^{-1}\frac{A_i}{P_{i+m}}$$

$$\begin{aligned} = (-)^2\frac{1}{p_1 p_2}\Bigg\{&\left(\frac{A_i}{P_{i+m}} + \frac{1}{p_1}\frac{A_{i+1}}{P_{i+m+1}} + \ldots \frac{1}{p_1^{n-i}}\frac{A_n}{P_{n+m+1}}\right) \\ &+ \frac{1}{p_2}\left(\frac{A_{i+1}}{P_{i+m+1}} + \frac{1}{p_1}\frac{A_{i+2}}{P_{i+m+2}} + \ldots \frac{1}{p_1^{n-i-1}}\frac{A_n}{P_{n+m+1}}\right) \\ &+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \Bigg\} \end{aligned}$$

$$= (-)^2\frac{1}{p_1 p_2}\left\{\frac{A_i}{P_{i+m}} + \left(\frac{1}{p_1}, \frac{1}{p_2}\right)\frac{A_{i+1}}{P_{i+m+1}} + \ldots \left(\frac{1}{p_1}, \frac{1}{p_2}\right)^{n-i}\frac{A_n}{P_{n+m+1}}\right\};$$

and consequently

$$a_{i-m}=(-)^{2m}\frac{1}{\mathrm{a}_{m,0}p_1p_2\ldots p_{2m}}\left\{\frac{\mathrm{A}_i}{\mathrm{P}_{i+m}}+\left(\frac{1}{p_1},\frac{1}{p_2}\ldots\frac{1}{p_{2m}}\right)\frac{\mathrm{A}_{i+1}}{\mathrm{P}_{i+m+1}}+\ldots\left(\frac{1}{p_1},\frac{1}{p_2}\ldots\frac{1}{p_{2m}}\right)^{n-i}\frac{\mathrm{A}_n}{\mathrm{P}_{n+m+1}}\right\}.$$

The expression $(\varepsilon^{\frac{d}{di}}-p_1)^{-1}(\varepsilon^{\frac{d}{di}}-p_2)^{-1}\ldots(\varepsilon^{\frac{d}{di}}-p_{2m})^{-1}$ may be resolved into its partial fractions in the usual way by writing

$$\mathrm{C}_1=\frac{p_1^{2m-1}}{(p_1-p_2)(p_1-p_3)\ldots(p_1-p_{2m})},\quad \mathrm{C}_2=\frac{p_2^{2m-1}}{(p_2-p_1)(p_2-p_3)\ldots(p_2-p_{2m})},\ldots$$

and then it takes the form

$$\frac{\mathrm{C}_1}{\varepsilon^{\frac{d}{di}}-p_1}+\frac{\mathrm{C}_2}{\varepsilon^{\frac{d}{di}}-p_2}+\ldots\frac{\mathrm{C}_{2m}}{\varepsilon^{\frac{d}{di}}-p_{2m}}$$

$$=\frac{\mathrm{C}_1}{p_1}+\frac{\mathrm{C}_2}{p_2}+\ldots+\left(\frac{\mathrm{C}_1}{p_1^2}+\frac{\mathrm{C}_2}{p_2^2}+\ldots\right)\varepsilon^{\frac{d}{di}}+\ldots,$$

giving

$$a_{i-m}=\frac{1}{\mathrm{a}_{m,0}}\left\{\left(\frac{\mathrm{C}_1}{p_1}+\frac{\mathrm{C}_2}{p_2}+\ldots\right)\frac{\mathrm{A}_i}{\mathrm{P}_{i+m}}+\left(\frac{\mathrm{C}_1}{p_1^2}+\frac{\mathrm{C}^2}{p_2^2}+\ldots\right)\frac{\mathrm{A}_{i+1}}{\mathrm{P}_{i+m+1}}+\ldots\left(\frac{\mathrm{C}_1}{p_1^{n-i+1}}+\frac{\mathrm{C}_2}{p_2^{n-i+1}}+\ldots\right)\frac{\mathrm{A}_n}{\mathrm{P}_{n+m+1}}\right\}.$$

§ 3. *The case of* s$_i$ *being any function of* x, y.

Although it seems doubtful, on account of their complexity, whether the following formulæ are likely to be of much practical use, it is still worth while to complete the theory by considering the most general case where s_i, instead of being linear, is any function whatever of x, y.

Since

$$s_1 s\frac{d^2}{dx^2}=s_1\frac{d}{dx}\cdot s\frac{d}{dx}-s_1\frac{ds}{dx}\frac{d}{dx}$$

$$=s\frac{d}{dx}\cdot s_1\frac{d}{dx}-s\frac{ds_1}{dx}\frac{d}{dx},$$

therefore

$$2s_1 s\frac{d^2}{dx^2}=\left(s_1\frac{d}{dx}-\frac{ds}{dx}\right)s\frac{d}{dx}+\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_1\frac{d}{dx}.$$

Again,

$$s_2 s_1 s\frac{d^3}{dx^3}=s\frac{d}{dx}\cdot s_2 s_1\frac{d^2}{dx^2}-s\frac{ds_2 s_1}{dx}\frac{d^2}{dx^2}$$

$$=s_1\frac{d}{dx}\cdot s s_2\frac{d^2}{dx^2}-s_1\frac{d\,s s_2}{dx}\frac{d^2}{dx^2}$$

$$=s_2\frac{d}{dx}\cdot s_1 s\frac{d^2}{dx^2}-s_2\frac{d\,s_1 s}{dx}\frac{d^2}{dx^2}.$$

Hence

$$
\begin{aligned}
6s_2s_1s\frac{d^3}{dx^3}=&\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)2s_2s_1\frac{d^2}{dx^2}\\
&+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)2s\,s_2\frac{d^2}{dx^2}\\
&+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)2s_1s\frac{d^2}{dx^2}
\end{aligned}
$$

$$
\begin{aligned}
=&\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s_2\frac{d}{dx}+\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s_1\frac{d}{dx}\\
&+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s\frac{d}{dx}+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_2\frac{d}{dx}\\
&+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_1\frac{d}{dx}+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s\frac{d}{dx}
\end{aligned}
$$

$$
\begin{aligned}
=&\,s\frac{d}{dx}s_1\frac{d}{dx}s_2\frac{d}{dx}-2\frac{ds}{dx}s_1\frac{d}{dx}s_2\frac{d}{dx}-\frac{ds_1}{dx}s\frac{d}{dx}s_2\frac{d}{dx}+2\frac{ds}{dx}\frac{ds_1}{dx}s_2\frac{d}{dx}-s\frac{d^2s_1}{dx^2}s_2\frac{d}{dx}\\
&+s\frac{d}{dx}s_2\frac{d}{dx}s_1\frac{d}{dx}-2\frac{ds}{dx}s_2\frac{d}{dx}s_1\frac{d}{dx}-\frac{ds_2}{dx}s\frac{d}{dx}s_1\frac{d}{dx}+2\frac{ds}{dx}\frac{ds_2}{dx}s_1\frac{d}{dx}-s\frac{d^2s_2}{dx^2}s_1\frac{d}{dx}\\
&+s_1\frac{d}{dx}s_2\frac{d}{dx}s\frac{d}{dx}-2\frac{ds_1}{dx}s_2\frac{d}{dx}s\frac{d}{dx}-\frac{ds_2}{dx}s_1\frac{d}{dx}s\frac{d}{dx}+2\frac{ds_1}{dx}\frac{ds_2}{dx}s\frac{d}{dx}-s_1\frac{d^2s_2}{dx^2}s\frac{d}{dx}\\
&+s_1\frac{d}{dx}s\frac{d}{dx}s_2\frac{d}{dx}-2\frac{ds_1}{dx}s\frac{d}{dx}s_2\frac{d}{dx}-\frac{ds}{dx}s_1\frac{d}{dx}s_2\frac{d}{dx}+2\frac{ds_1}{dx}\frac{ds}{dx}s_2\frac{d}{dx}-s_1\frac{d^2s}{dx^2}s_2\frac{d}{dx}\\
&+s_2\frac{d}{dx}s\frac{d}{dx}s_1\frac{d}{dx}-2\frac{ds_2}{dx}s\frac{d}{dx}s_1\frac{d}{dx}-\frac{ds}{dx}s_2\frac{d}{dx}s_1\frac{d}{dx}+2\frac{ds_2}{dx}\frac{ds}{dx}s_1\frac{d}{dx}-s_2\frac{d^2s}{dx^2}s_1\frac{d}{dx}\\
&+s_2\frac{d}{dx}s_1\frac{d}{dx}s\frac{d}{dx}-2\frac{ds_2}{dx}s_1\frac{d}{dx}s\frac{d}{dx}-\frac{ds_2}{dx}s_1\frac{d}{dx}s\frac{d}{dx}+2\frac{ds_2}{dx}\frac{ds_1}{dx}s\frac{d}{dx}-s_2\frac{d^2s_1}{dx^2}s\frac{d}{dx}
\end{aligned}
$$

$$
\begin{aligned}
=&\,s\frac{d}{dx}s_1\frac{d}{dx}s_2\frac{d}{dx}+s\frac{d}{dx}s_2\frac{d}{dx}s_1\frac{d}{dx}+s_1\frac{d}{dx}s_2\frac{d}{dx}s\frac{d}{dx}+s_1\frac{d}{dx}s\frac{d}{dx}s_2\frac{d}{dx}+s_2\frac{d}{dx}s\frac{d}{dx}s_1\frac{d}{dx}+s_2\frac{d}{dx}s_1\frac{d}{dx}s\frac{d}{dx}\\
&-3\frac{ds}{dx}\left(s_1\frac{d}{dx}s_2\frac{d}{dx}+s_2\frac{d}{dx}s_1\frac{d}{dx}\right)-3\frac{ds_1}{dx}\left(s_2\frac{d}{dx}s\frac{d}{dx}+s\frac{d}{dx}s_2\frac{d}{dx}\right)-3\frac{ds_2}{dx}\left(s\frac{d}{dx}s_1\frac{d}{dx}+s_1\frac{d}{dx}s\frac{d}{dx}\right)\\
&+\left(4\frac{ds_1}{dx}\frac{ds_2}{dx}-s_1\frac{d^2s_2}{dx^2}-s_2\frac{d^2s_1}{dx^2}\right)s\frac{d}{dx}\\
&+\left(4\frac{ds_2}{dx}\frac{ds}{dx}-s_2\frac{d^2s}{dx^2}-s\frac{d^2s_2}{dx^2}\right)s_1\frac{d}{dx}\\
&+\left(4\frac{ds}{dx}\frac{ds_1}{dx}-s_1\frac{d^2s_1}{dx^2}-s_1\frac{d^2s}{dx^2}\right)s_2\frac{d}{dx}.
\end{aligned}
$$

Suppose that the above law holds good for m factors; then

$$1.2\,..\,m s s_1\,..\,s_m\frac{d^m}{dx^m}=\Sigma\left(s\frac{d}{dx}-(m-1)\frac{ds}{dx}\right)\left(s_1\frac{d}{dx}-(m-2)\frac{ds_1}{dx}\right)..\,s_{m-1}\frac{d}{dx},$$

where Σ indicates that the sum of all products formed by the interchanges of the suffixes $0, 1, ..\, m-1$, is to be taken. Then

$$1.2\ldots(m+1)ss_1\ldots s_m\frac{d^{m+1}}{dx^{m+1}}$$

$$=s\frac{d}{dx}1.2\ldots ms_1s_2\ldots s_m\frac{d^m}{dx^m}-1.2\ldots ms\frac{ds_1s_2\ldots s_m}{dx}\frac{d^m}{dx^m}$$

$$+s_1\frac{d}{dx}1.2\ldots m\,s\,s_2\ldots s_m\frac{d^m}{dx^m}-1.2\ldots ms_1\frac{dss_2\ldots s_m}{dx}\frac{d^m}{dx^m}$$

$$+\ldots$$

$$=\left(s\frac{d}{dx}-m\frac{ds}{dx}\right)1.2\ldots ms_1s_2\ldots s_m\frac{d^m}{dx^m}$$

$$+\left(s_1\frac{d}{dx}-m\frac{ds_1}{dx}\right)1.2\ldots m\,s\,s_2\ldots s_m\frac{d^m}{dx^m}$$

$$+\ldots$$

$$=\Sigma\left(s\frac{d}{dx}-m\frac{ds}{dx}\right)\left(s_1\frac{d}{dx}-(m-1)\frac{ds_1}{dx}\right)\ldots s_m\frac{d}{dx},$$

which proves it in the general case.

Again,

$$ss_1\frac{d^2}{dx\,dy}=s\frac{d}{dx}s_1\frac{d}{dy}-s\frac{ds_1}{dx}\frac{d}{dy}$$

$$=s\frac{d}{dy}s_1\frac{d}{dx}-s\frac{ds_1}{dy}\frac{d}{dx}$$

$$=s_1\frac{d}{dx}s\frac{d}{dy}-s_1\frac{ds}{dx}\frac{d}{dy}$$

$$=s_1\frac{d}{dy}s\frac{d}{dx}-s_1\frac{ds}{dy}\frac{d}{dx}.$$

Hence

$$2ss_1\frac{d^2}{dx\,dy}=\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_1\frac{d}{dy}+\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s\frac{d}{dy}$$

$$=\left(s\frac{d}{dy}-\frac{ds}{dy}\right)s_1\frac{d}{dx}+\left(s_1\frac{d}{dy}-\frac{ds_1}{dy}\right)s\frac{d}{dx}.$$

Again,

$$ss_1s_2\frac{d^3}{dx^2dy}=s\frac{d}{dy}s_1s_2\frac{d^2}{dx^2}-\frac{ds_1}{dy}s_2s\frac{d^2}{dx^2}-\frac{ds_2}{dy}ss_1\frac{d^2}{dx^2}$$

$$=-\frac{ds}{dy}s_1s_2\frac{d^2}{dx^2}+s_1\frac{d}{dy}s_2s\frac{d^2}{dx^2}-\frac{ds_2}{dy}ss_1\frac{d^2}{dx^2}$$

$$=-\frac{ds}{dy}s_1s_2\frac{d^2}{dx^2}-\frac{ds_1}{dy}s_2s\frac{d^2}{dx^2}+s_2\frac{d}{dy}ss_1\frac{d^2}{dx^2}.$$

Hence

$$6ss_1s_2\frac{d^3}{dx^2\,dy}=\left(s\frac{d}{dy}-2\frac{ds}{dy}\right)2s_1s_2\frac{d^2}{dx^2}$$

$$+\left(s_1\frac{d}{dy}-2\frac{ds_1}{dy}\right)2s_2\,s\frac{d^2}{dx^2}$$

$$+\left(s_2\frac{d}{dy}-2\frac{ds_2}{dy}\right)2s\,s_1\frac{d^2}{dx^2}.$$

Again,

$$ss_1s_2\frac{d^3}{dx^2\,dy}= s\frac{d}{dx}s_1s_2\frac{d^2}{dx\,dy}-\frac{ds_1}{dx}s_2s\frac{d^2}{dx\,dy}-\frac{ds_2}{dx}ss\frac{d^2}{dx\,dy}$$

$$=-\frac{ds}{dx}s_1s_2\frac{d^2}{dx\,dy}+s_1\frac{d}{dx}s_2s\frac{d^2}{dx\,dy}-\frac{ds_2}{dx}ss_1\frac{d^2}{dx\,dy}$$

$$=-\frac{ds}{dx}s_1s_2\frac{d^2}{dx\,dy}-\frac{ds_1}{dx}s_2s\frac{d^2}{dx\,dy}+s_2\frac{d}{dx}ss_1\frac{d^2}{dx\,dy},$$

whence

$$6ss_1s_2\frac{d^3}{dx^2\,dy}=\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)2s_1s_2\frac{d^2}{dx\,dy}$$

$$+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)2s_2s\frac{d^2}{dx\,dy}$$

$$+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)2ss_1\frac{d^2}{dx\,dy};$$

and consequently the expression $6ss_1s_2\frac{d^3}{dx^2dy}$ admits of three equivalents, as follows:

$$6ss_1s_2\frac{d^3}{dx^2\,dy}$$

$$=\left(s\frac{d}{dy}-2\frac{ds}{dy}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s_2\frac{d}{dx}+\left(s\frac{d}{dy}-2\frac{ds}{dy}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s\frac{d}{dx}$$

$$+\left(s_1\frac{d}{dy}-2\frac{ds_1}{dy}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s\frac{d}{dx}+\left(s_1\frac{d}{dy}-2\frac{ds_1}{dy}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_2\frac{d}{dx}$$

$$+\left(s_2\frac{d}{dy}-2\frac{ds_2}{dy}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_1\frac{d}{dx}+\left(s_2\frac{d}{dy}-2\frac{ds_2}{dy}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s\frac{d}{dx}$$

$$=\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_1\frac{d}{dy}-\frac{ds_1}{dy}\right)s_2\frac{d}{dx}+\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_2\frac{d}{dy}-\frac{ds_2}{dy}\right)s_1\frac{d}{dx}$$

$$+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s_2\frac{d}{dy}-\frac{ds_2}{dy}\right)s\frac{d}{dx}+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s\frac{d}{dy}-\frac{ds}{dy}\right)s_2\frac{d}{dx}$$

$$+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s\frac{d}{dy}-\frac{ds}{dy}\right)s_1\frac{d}{dx}+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s_1\frac{d}{dy}-\frac{ds_1}{dy}\right)s\frac{d}{dx}$$

$$=\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s_2\frac{d}{dy}+\left(s\frac{d}{dx}-2\frac{ds}{dx}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s_1\frac{d}{dy}$$

$$+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s_2\frac{d}{dx}-\frac{ds_2}{dx}\right)s\frac{d}{dy}+\left(s_1\frac{d}{dx}-2\frac{ds_1}{dx}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_2\frac{d}{dy}$$

$$+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s\frac{d}{dx}-\frac{ds}{dx}\right)s_1\frac{d}{dy}+\left(s_2\frac{d}{dx}-2\frac{ds_2}{dx}\right)\left(s_1\frac{d}{dx}-\frac{ds_1}{dx}\right)s\frac{d}{dy}.$$

In short, in order to change the expression for $6ss_1s_2\frac{d^3}{dx^3}$ into that for $6ss_1s_2\frac{d^3}{dx^2\,dy}$, we have only to write $\frac{d}{dy}$ for $\frac{d}{dx}$ in *one* of the factors of each term; the *same* factor being changed throughout.

Writing x for y, and y for x, we have the corresponding expression for $6ss_1s_2\frac{d^3}{dx\,dy^2}$. And a train of reasoning similar to that used in calculating $1.2\,..(m+1)ss_1\,..\,s_m\frac{d^{m+1}}{dx^{m+1}}$ will give

$$1.2\,..(m+1)ss_1\,..\,s_m\frac{d^{m+1}}{dx^{m+1-p}\,dy^p}$$

$$=\Sigma\left(s\frac{d}{dx}-(m+1)\frac{ds}{dx}\right)..\left(s_{m-p}\frac{d}{dx}-(m-p+1)\frac{ds_{m-p}}{dx}\right)\left(s_{m-p+1}\frac{d}{dy}-(m-p)\frac{ds_{m-p+1}}{dy}\right)..\,s_m\frac{d}{dy},$$

the y factors being detected at pleasure, provided that they are p in number, and the same factors throughout.

Postscript.

Received March 3, 1860.

In continuing my researches on the extended form of the Index Symbol, I have established a variety of formulæ relative to the case of many variables analogous to those in the case of two. As the expressions occurring in the investigations are frequently of great length, I propose here to give only the principal results, without entering into the details of proof.

Let $i_1, i_2, ..$ be any permutation of the series 1, 2, ..; and let it be symbolically represented thus:

$$i_1, i_2, .. = P_i(1, 2, ..).$$

Then any permutation P_j performed on P_i may be similarly represented by $j_{i_1}, j_{i_2}, ..$, or more simply, $ji_1, ji_2, ..$ Then $ji_1, ji_2, ..$ will represent a new permutation of the original series 1, 2, ..; and the notation above adopted may be extended thus:

$$ji_1, ji_2, .. = P_{ji}(1, 2, ..)$$

$$P_j(i_1, i_2, ..) = P_jP_i(1, 2, ..),$$

or, dropping the subject of operation,

$$P_{ji} = P_jP_i,$$

and so on generally.

Let there be any number of variables $x_1, x_2, \ldots$, and let

$$\nabla_i = x_{i_1}\frac{d}{dx_1} + x_{i_2}\frac{d}{dx_2} + \ldots$$

$$\nabla_j = x_{j_1}\frac{d}{dx_1} + x_{j_2}\frac{d}{dx_2} + \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and D_i, D_j, .. corresponding expressions when the $\frac{d}{dx_1}, \frac{d}{dx_2}, \ldots$ operate only on some extraneous subject of differentiation, and not on $x_1, x_2, \ldots$ so far as they appear explicitly in the values of D_i, D_j, ..

Then it will be found that

$$D_j D_i = \nabla_j \nabla_i - \nabla_{ji}$$

$$D_k D_j D_i = \nabla_j \nabla_k \nabla_i - \nabla_{kj} \nabla_i - \nabla_j \nabla_{ki} - \nabla_k \nabla_{ji} + \nabla_{kji} + \nabla_{jki};$$

and so on for any number of Ds and ∇s. There are special cases in which these expressions take a more symmetrical form. Thus, if the second condition of the system

$$P(j, k) = P(k, j), \quad P(k, i) = P(i, k), \quad P(i, j) = P(j, i)$$

be satisfied, the group ∇_{kj}, ∇_{ki}, ∇_{ji} may be replaced by the group ∇_{kj}, ∇_{ik}, ∇_{ji}, in which the suffixes are cyclically arranged. If the first and last are satisfied, the group may be replaced by ∇_{ik}, ∇_{ki}, ∇_{ij}. If, besides, the first condition of the system

$$P\{i(j, k)\} = P\{(j, k)i\}$$

$$P\{j(k, i)\} = P\{(k, i)j\}$$

$$P\{k(i, j)\} = P\{(i, j)k\}$$

be satisfied, the expression for $D_k D_j D_i$ may be written

$$\nabla_k \nabla_j \nabla_i - \nabla_i \nabla_{kj} - \nabla_j \nabla_{ik} - \nabla_k \nabla_{ji} + 2\nabla_{kji},$$

or

$$\nabla_k \nabla_j \nabla_i - \nabla_i \nabla_{jk} - \nabla_j \nabla_{ki} - \nabla_k \nabla_{ij} + 2\nabla_{kji},$$

according as the second, or the first and last conditions of the first system are satisfied. And similarly, if the two last conditions of the second system are satisfied, the expression may be written

$$\nabla_k \nabla_j \nabla_i - \nabla_{kj} \nabla_i - \nabla_{ik} \nabla_j - \nabla_{ji} \nabla_k + 2\nabla_{jik}$$

$$\nabla_k \nabla_j \nabla_i - \nabla_{jk} \nabla_i - \nabla_{ki} \nabla_j - \nabla_{ij} \nabla_k + 2\nabla_{jik}.$$

If the accents in the symbols ∇'_i, ∇'_j, .. are understood to imply that the suffixes $i, j, \ldots$ and not the ∇s are to be combined, *e. g.*

$$(a, b, c, d \between \nabla'_j \nabla'_i)^3$$

$$= a\nabla_{j^3} + b(\nabla_{j^2i} + \nabla_{jij} + \nabla_{ij^2}) + c(\nabla_{ji^2} + \nabla_{iji} + \nabla_{ji^2}) + d\nabla_{i^3},$$

then

$$(a,\ b,\ c,\ d \between D_j,\ D_i)^3$$

$$=(a,b,c,d \between \nabla_j, \nabla_i)^3 - 2(a,b,c,d \between \nabla_j, \nabla_i \between \nabla'_j, \nabla'_i)^2 - (a,b,c,d \between \nabla'_j, \nabla'_i \overset{2}{\between} \nabla_j, \nabla_i) + 2(a,b,c,d \between \nabla'_j, \nabla'_i)^3,$$

which may be symbolically represented thus:

$$(a,\ b,\ c,\ d \between D_j,\ \nabla_i)^3 = (a,\ b,\ c,\ d \between \left|\begin{matrix} (\nabla_j\ \nabla_i) & 1 & * \\ (\nabla'_j\ \nabla'_i)^2 & (\nabla_j\ \nabla_i) & 2 \\ (\nabla'_j\ \nabla'_i)^3 & (\nabla'_j\ \nabla'_i)^2 & (\nabla_j\ \nabla_i) \end{matrix}\right|\ )$$

And generally,

$$(a,\ b, .. \between .. D_j,\ \nabla_i)^n = (a,\ b, .. \between \left|\begin{matrix} (..\nabla_j\ \nabla_i) & 1 & * & .. & * \\ (..\nabla'_j\ \nabla'_i)^2 & (..\nabla_j\ \nabla_i) & 2 & .. & * \\ (..\nabla'_j\ \nabla'_i)^3 & (..\nabla'_j\ \nabla'_i)^2 & (..\nabla_j\ \nabla_i) & .. & * \\ .. & .. & .. & .. & .. \\ (..\nabla'_j\ \nabla'_i)^n & (..\nabla'_j\ \nabla'_i)^{n-1} & (..\nabla'_j\ \nabla'_i)^{n-2} & .. & (..\nabla_j\ \nabla_i) \end{matrix}\right|\ )$$

Moreover, if

$$u = \Sigma(1^{\alpha_1}\, 2^{\alpha_2} ..)x_1^{\alpha_1}\, x_2^{\alpha_2} ..$$

Then, understanding that the order of multiplication must be preserved, as in the case of two variables,

$$F(..\nabla_j\nabla_i)u = \Sigma F\left\{..,\ \Sigma(\alpha_s+1)\epsilon^{\frac{d}{d\alpha_s}-\frac{d}{d\alpha_{j_s}}},\quad \Sigma(\alpha_r+1)\epsilon^{\frac{d}{d\alpha_r}-\frac{d}{d\alpha_{i_r}}}\right\}(1^{\alpha_1}\, 2^{\alpha_2} ..)x_1^{\alpha_1}\, x_2^{\alpha_2} ..$$

Again, if

$$u_{11} = \alpha_{11}x_1 + \beta_{11}x_2 + .. \qquad u_{12} = \alpha_{12}x_1 + \beta_{12}x_2 ..$$

$$u_{21} = \alpha_{21}x_1 + \beta_{21}x_2 + .. \qquad u_{22} = \alpha_{22}x_1 + \beta_{22}x_2 ..$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad .$$

$$\nabla_1 = u_{11}\frac{d}{dx_1} + u_{12}\frac{d}{dx_2} + ..$$

$$\nabla_2 = u_{21}\frac{d}{dx_1} + u_{22}\frac{d}{dx_2} + ..$$

$$. \quad . \quad . \quad . \qquad . \quad . \quad .$$

Then

$$\begin{aligned} D_2D_1 = \nabla_2\nabla_1 &- \left\{(\alpha_{11}\alpha_{21} + \beta_{11}\alpha_{22} + ..)x_1 + (\alpha_{11}\beta_{21} + \beta_{11}\beta_{22} + ..)x_2 + ..\right\}\frac{d}{dx_1} \\ &- \left\{(\alpha_{12}\alpha_{21} + \beta_{12}\alpha_{22} + ..)x_1 + (\alpha_{12}\beta_{21} + \beta_{12}\beta_{22} + ..)x_2 + ..\right\}\frac{d}{dx_2} \\ &- \ldots \end{aligned}$$

Now the coefficients of the xs are all comprised in the product

$$\left|\begin{array}{ccc} \alpha_{21} & \alpha_{22} & .. \\ \beta_{21} & \beta_{22} & .. \\ . & . & .. \end{array}\right\|\begin{array}{ccc} \alpha_{11} & \beta_{11} & .. \\ \alpha_{12} & \beta_{12} & .. \\ . & . & .. \end{array}\right|$$

and the terms correspond line for line and column for column. If, therefore,

$$\left(\begin{array}{ccc} a & b & .. \\ a_1 & b_1 & .. \\ . & . & .. \end{array}\right)\!\!\left(x, x_1 .. \right)\!\!\left(y, y_1 .. \right) = (a\,x+b\,x_1+..)y \\ +(a_1x+b_1x_1+..)y_1 \\ +\ldots$$

we may write

$$D_2D_1=\nabla_2\nabla_1-\left(\begin{array}{ccc} \alpha_{21} & \beta_{22} & .. \\ \beta_{21} & \beta_{22} & .. \\ . & . & .. \end{array}\right\|\begin{array}{ccc} \alpha_{11} & \beta_{11} & .. \\ \alpha_{12} & \beta_{12} & .. \\ . & . & .. \end{array}\right)\!\!\left(x_1, x_2, .. \right)\!\!\left(\frac{d}{dx_1}, \frac{d}{dx_2}, .. \right)$$

and by a tolerably obvious extension of the symbolical notation used in other parts of this paper, we may write this expression thus:

$$\nabla_2\nabla_1-\nabla_2'\nabla_1'.$$

And generally

$$D_nD_{n-1}..D_2\nabla_1=\left|\begin{array}{cccc} \nabla_n & \nabla_{n-1}' & .. & \nabla_1' \\ \nabla_n' & \nabla_{n-1} & .. & \nabla_1' \\ . & . & .. & . \\ \nabla_n' & \nabla_{n-1}' & .. & \nabla_1 \end{array}\right|$$

Lastly, the formulæ for $ss_1 .. \frac{d^n}{dx^{n-p}dy^p}$ in the case of two variables may be directly extended to the corresponding case of $ss_1 .. \frac{d^n}{dx_1^\alpha dx_2^\beta ..}$ with many variables.